Name _Solutions____ Linear Algebra, Quiz 5

1) Using the information below, find a formula for $\vec{x}_{B_2}$. That is, find the vector $\vec{x}$ expressed in B_2 coordinates. No need to simplify anything.

$$B_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}, B_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \vec{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}_{B_1}$$

$$\begin{bmatrix} 1 & 7 & 1 \\ 2 & 6 & 0 \\ 3 & 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 & 4 \\ 1 & 5 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}_{B_1}$$

2) A linear operator T takes input from $\mathbb{R}^7$. It is known that there is a vector $\vec{b}$ such that $T(\vec{x}) = \vec{b}$ has at least 4 solutions. List all possible values for $\dim(\mathrm{Col}([T]))$. That is, list all possible values for the dimension of the column space of the matrix associated to T.

This is a rank-nullity theorem problem. We just need to figure out what's what:

Rank + nullity = number of columns

The fact that T takes input from $\mathbb{R}^7$ tells us that the number of columns in $[T]$ is 7:

Rank + nullity = 7

The fact that $T(\vec{x}) = \vec{b}$ tells us that $\ker(T)$ is nontrivial, that is, the nullity of $[T]$ is at least 1. (Not at least 4, because we don't know that the solutions are all linearly independent. There might only be one free variable. One free variable means they'll be infinitely many solutions when consistent, so the number "4" was a red herring: effectively we were told that $T(\vec{x}) = \vec{b}$ has infinitely many solutions)

Rank + [something at least 1] = 7

We're not told anything else, so the rank could be anything that satisfies the above equation. These values are:

0, 1, 2, 3, 4, 5, 6