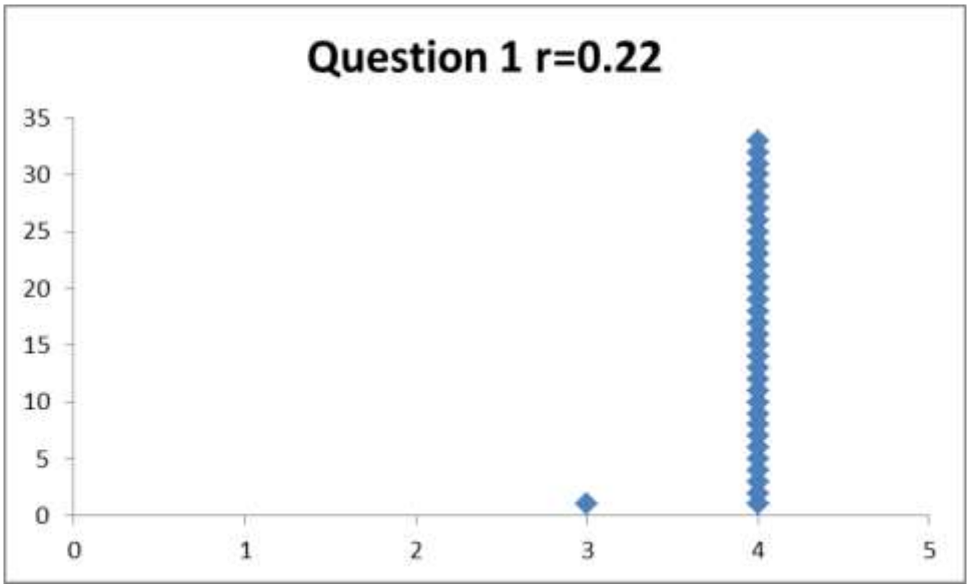
Name ____Solutions____ Linear Algebra, Test 1, 9/22/2014

1) Write the vector equation below as a system of linear equations. (4 points)

$$\begin{bmatrix}2\\3\\4\end{bmatrix}x_1 + \begin{bmatrix}1\\2\\3\end{bmatrix}x_2 = \begin{bmatrix}5\\6\\7\end{bmatrix}$$

$$\begin{aligned} 2x_1 + x_2 &= 5 \\ 3x_2 + 2x_2 &= 6 \\ 4x_1 + 3x_2 &= 7 \end{aligned}$$

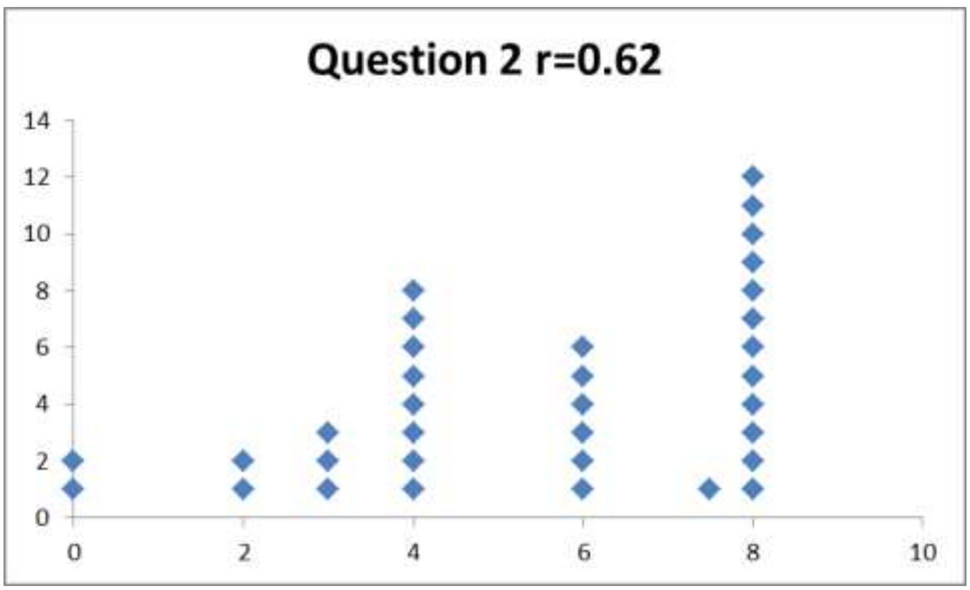


2) Determine whether or not the set below is a subspace of $\mathbb{R}^5$. If it is, find its dimension. Justify your answer. (8 points)

$$\left\{ \begin{bmatrix} a \\ a+b \\ 1+a+b+c \\ 0 \\ 1 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

No. To be a space it must satisfy all three properties, but it does not. Actually fails all three properties:

It does not contain zero (the 5th component is always 1)
Addition is not closed (the 5th component would not be 1)
Scalar multiplication is not closed (the 5th component would not be 1)

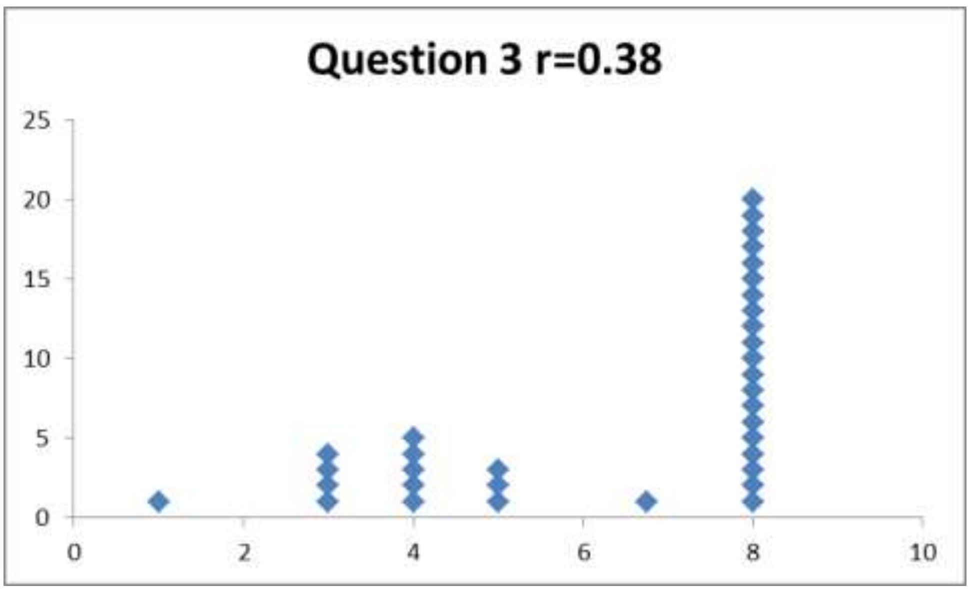


3) A linear transformation is given below. Determine whether or not it is onto. Justify your answer. (8 points)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ 4x_2 \\ x_1 + 3x_3 \\ x_1 + x_2 + x_3 \\ 6x_2 + 7x_3 \end{bmatrix}$$

This function maps from $\mathbb{R}^3$ to $\mathbb{R}^5$, so there's no possible way it could be onto: the range is at most a 3 dimensional subspace of $\mathbb{R}^5$.

(Also, if you row reduced it, you should find that there is at least one zero row – that is, a row without a pivot. Actually there are multiple.)

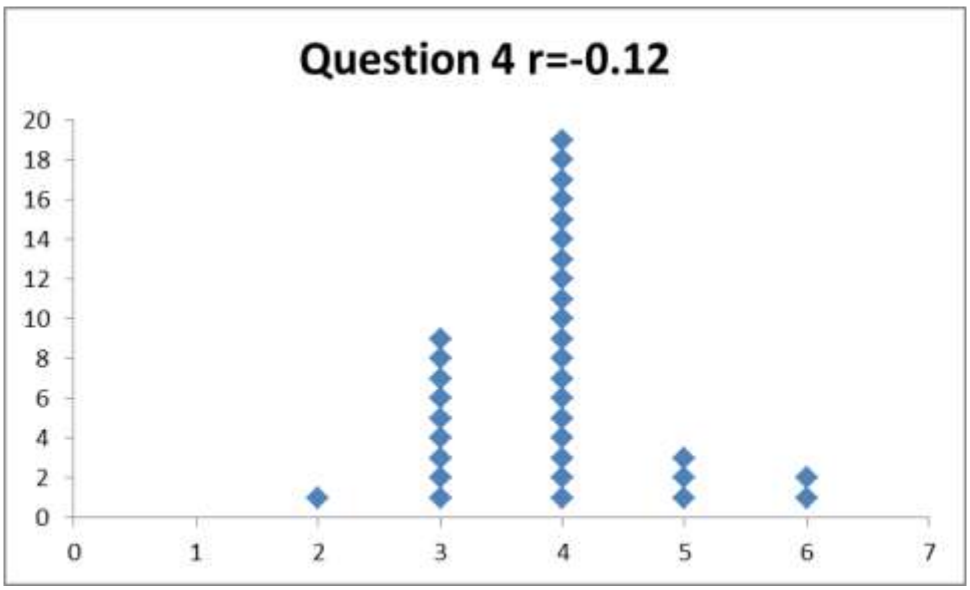


4) Determine which of the spaces below have dimension 2. Circle them. (6 points)

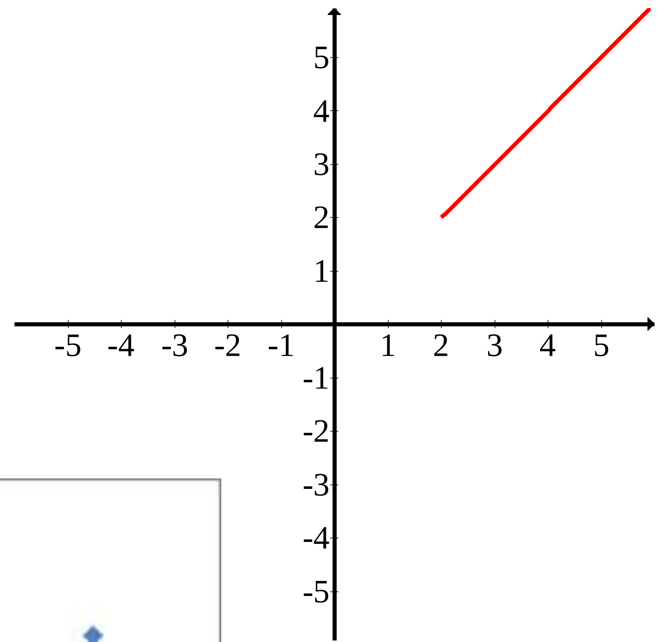
$\operatorname{span}\left(\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right\}\right)$ $\operatorname{span}\left(\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\4\end{bmatrix}\right\}\right)$ $\operatorname{span}\left(\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}1\\4\end{bmatrix}\right\}\right)$

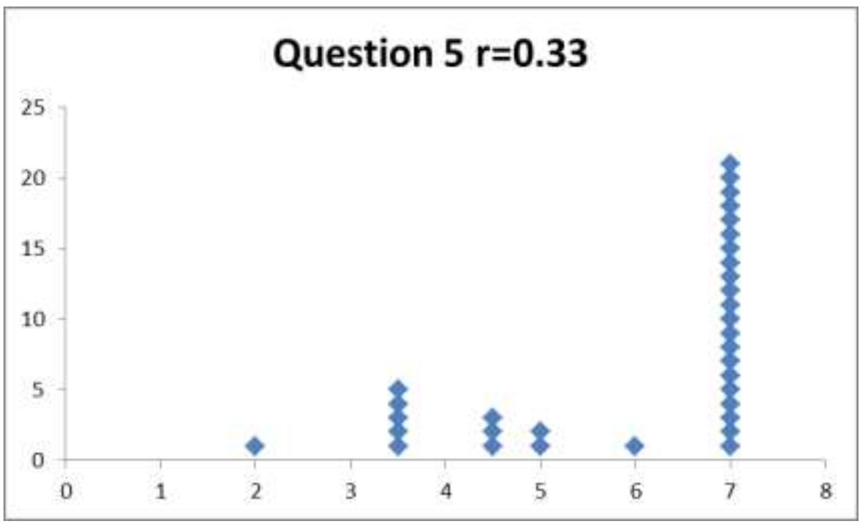
$\operatorname{span}\left(\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}2\\4\\6\end{bmatrix}\right\}\right)$ $\operatorname{span}\left(\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}2\\3\\4\end{bmatrix}\right\}\right)$ $\operatorname{span}\left(\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}2\\3\\4\end{bmatrix},\begin{bmatrix}3\\5\\7\end{bmatrix}\right\}\right)$

Pink marks: grading corrections. span $\left(\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\4\end{bmatrix}\right\}\right)$ was supposed to be span $\left(\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\4\end{bmatrix}\right\}\right)$…

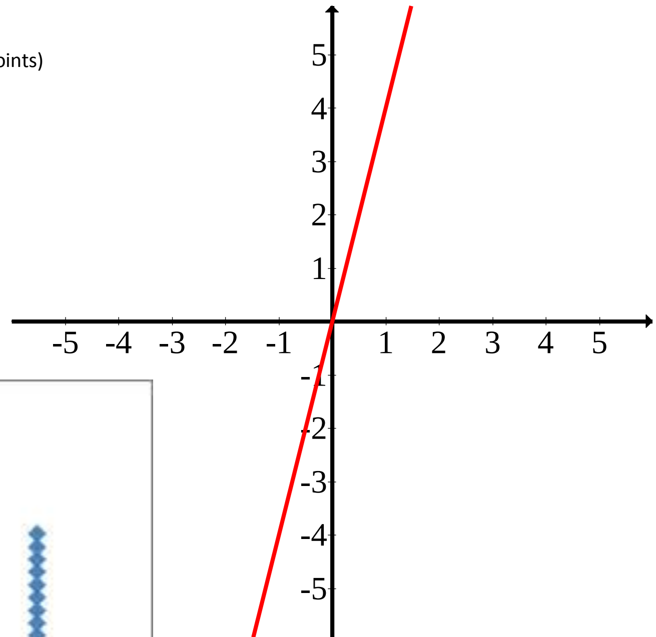


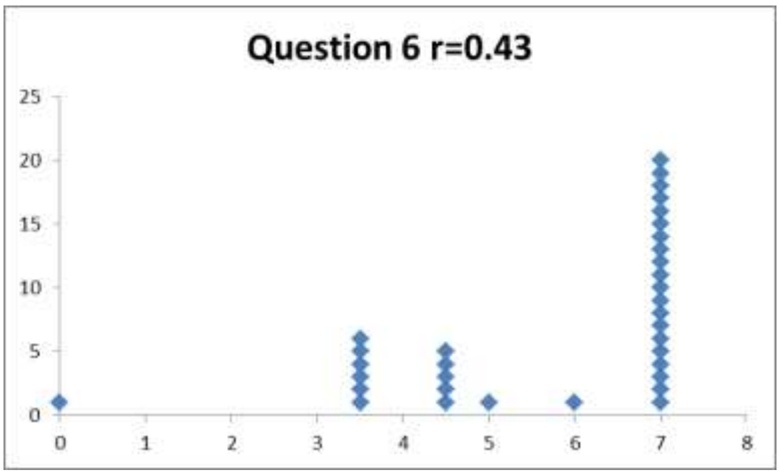
5) Illustrate $\left\{\begin{bmatrix} s \\ s \end{bmatrix} : s \geq 2\right\}$ on the axis below. (7 points)





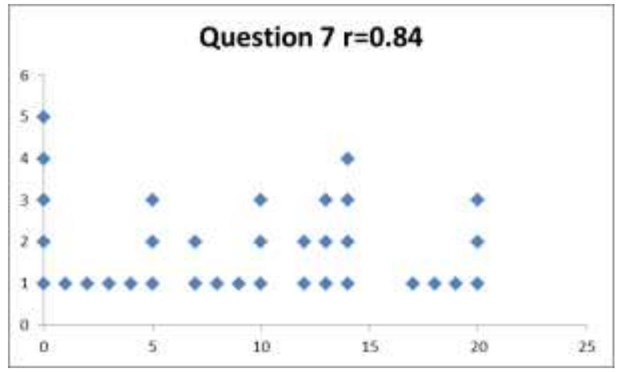
6) Illustrate span $\left(\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\8\end{bmatrix}\right\}\right)$ on the axis below. (7 points)





7) You know that the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^7$ is one-to-one. What else can you say?

(4 points per insightful statement; 1 point per obvious statement; every incorrect statement nullifies a correct statement. 20 points maximum)



Insightful statements:

The kernel of T is $\{\vec{0}\}$

The null space of $[T]$ is $\{\vec{0}\}$

Every column of $[T]$ has a pivot

$[T]$ has full row span.

The system of equations has no free variables.

The columns of $[T]$ are linearly independent

The columns of $[T]$ form a basis [for a 3 dimensional subspace of $\mathbb{R}^7$]

The equation $A\vec{x} = \vec{0}$ has a unique solution

The corresponding system of homogeneous equations has a unique solution

The corresponding homogeneous vector equation has a unique solution

For any vector $\vec{b} \in \mathbb{R}^7$, the equation $A\vec{x} = \vec{b}$ has at most one solution.

Every corresponding system of nonhomogeneous equations has at most one solution.

Every corresponding nonhomogeneous vector equation has at most one solution.

The linked statements are obviously equivalent: hence the first one is worth 4 points, with the other(s) worth 1 point.

Obvious statements:

The domain is $\mathbb{R}^3$

The codomain is $\mathbb{R}^7$

T is not onto

[Insert every equivalent statement to T not being onto here – there are perhaps a dozen of them?]

$[T]$ has 7 rows and 3 columns

Grading colors:

Red check – correct insightful statement

Red X – incorrect insightful statement

Blue check – correct obvious statement

Blue X - incorrect obvious statement

Green dash – I could not make sense of what you said. You should ask me or a classmate why it doesn't make sense.

Please express what you mean! Many people didn't get points for things because the expression didn't make sense, such as:

- Is linearly independent
- Null space of T is $\vec{0}$
- $\ker(T) = \vec{0}$
- Has no solutions

For the problems on this page, use the basis for $\mathbb{R}^2$ below. Do not change the order of the vectors.

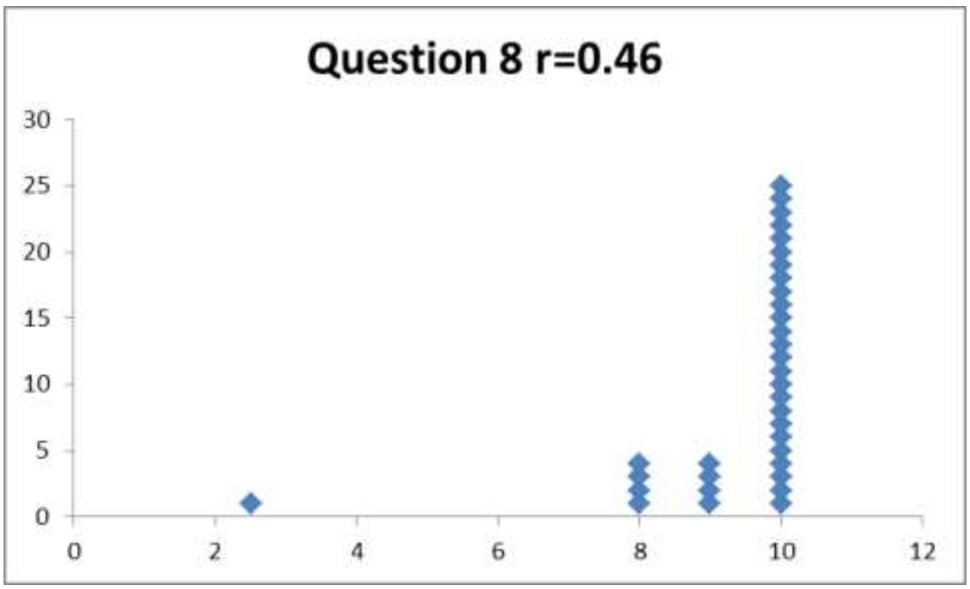
$$\mathcal{B} = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$$

8) Find the vector $\begin{bmatrix}4\\10\end{bmatrix}$ in terms of $\mathcal{B}$. (10 points)

We must solve $\begin{bmatrix}1\\2\end{bmatrix}x_1 + \begin{bmatrix}1\\0\end{bmatrix}x_2 = \begin{bmatrix}4\\10\end{bmatrix}$. By inspection we see that $x_1 = 5$ and $x_2 = -1$, hence:

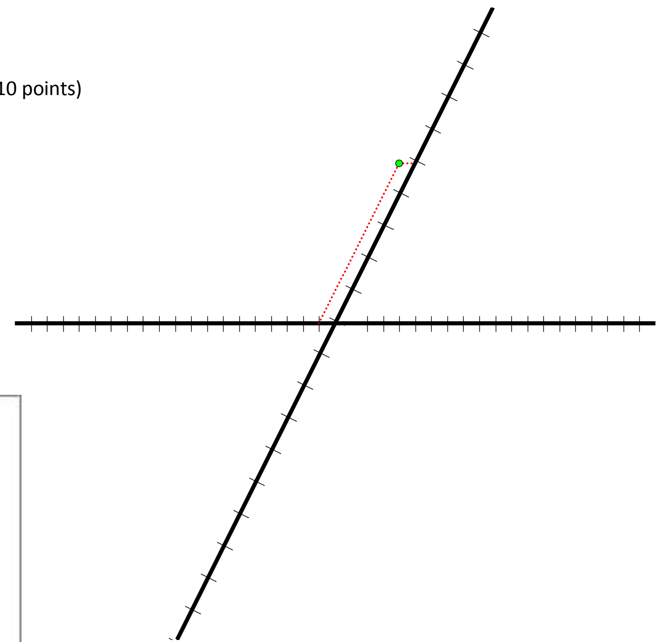
$$\begin{bmatrix}4\\10\end{bmatrix} = 5\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}$$

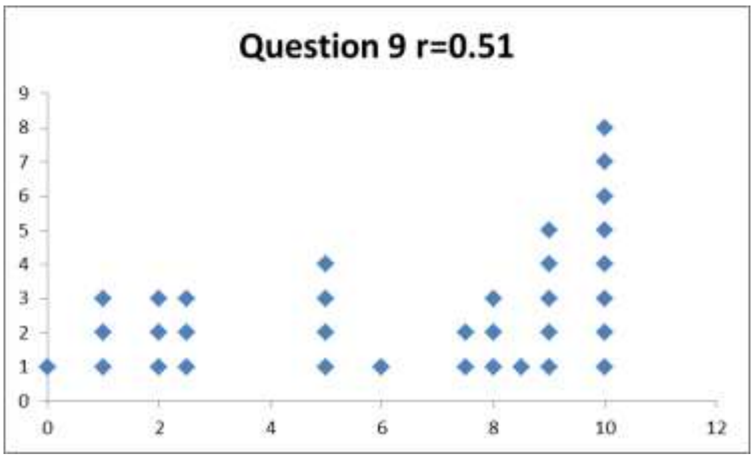
We'll develop notation for this sort of thing in the future. For now, liberal partial credit (often full credit) was given for questionable notation as long as an understanding of the underlying process was clear.



9) Graphically illustrate your answer to the above question. (10 points)

$$\begin{bmatrix} 4 \\ 10 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



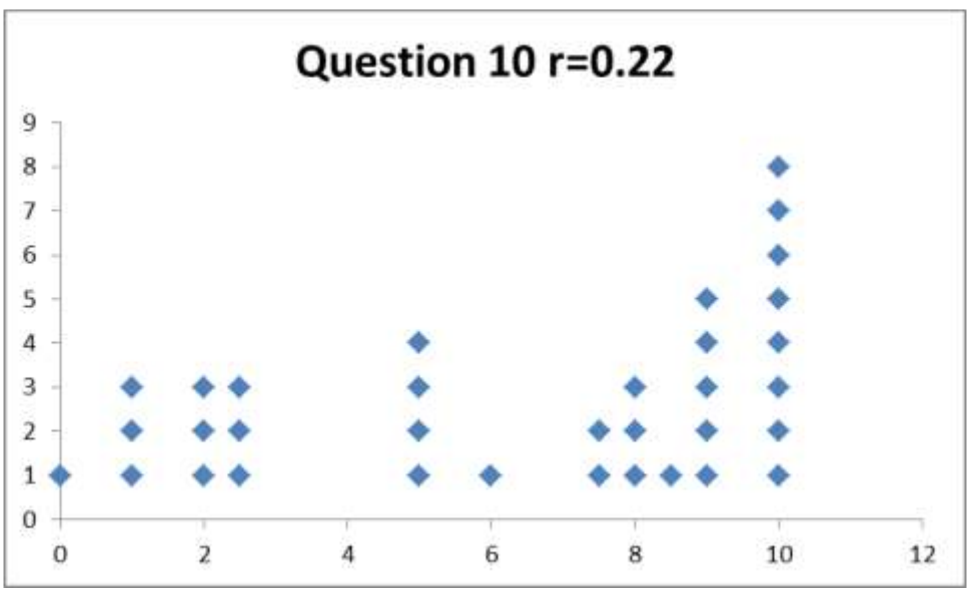


Use the matrix below for the problems on this page.

$$A = \begin{bmatrix} 3 & 6 & 3 & 12 \\ 1 & 2 & 2 & 7 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

10) Find the row reduced echelon form of A. (7 points)

$$\begin{bmatrix} 3 & 6 & 3 & 12 \\ 1 & 2 & 2 & 7 \\ 0 & 0 & 2 & 6 \end{bmatrix} \overset{R_1 \to \frac{1}{3}R_1}{\sim} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 2 & 2 & 7 \\ 0 & 0 & 2 & 6 \end{bmatrix} \underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix} \overset{R_3 \to 2R_2}{\sim} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_1 \to R_1 - R_2}{\sim} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

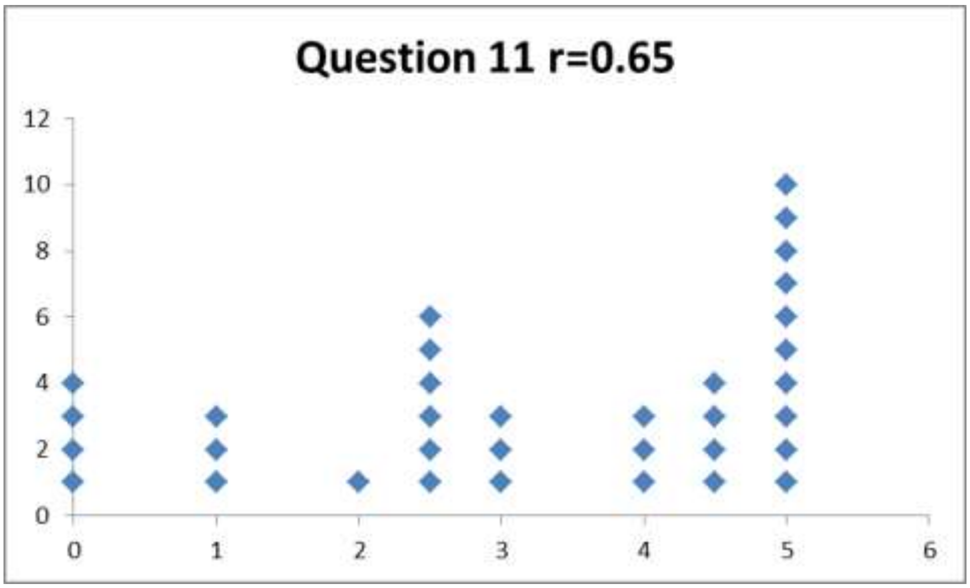


11) Find the null space of A. (5 points)

Solving $\begin{bmatrix}1 & 2 & 0 & 1\\0 & 0 & 1 & 3\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, we see that x_2 and x_4 are free. Solving for x_3 we get $x_3=-3x_4$.

Solving for x_1 we get $x_1=-2x_2-x_4$. Hence:

$$\text{null}(A)=\text{span}\left(\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-3\\1\end{bmatrix}\right\}\right)=\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}x_2+\begin{bmatrix}-1\\0\\-3\\1\end{bmatrix}x_4 : x_2,x_4\in\mathbb{R}\right\}=\left\{\begin{bmatrix}-2x_2-x_4\\x_2\\-3x_4\\x_4\end{bmatrix} : x_2,x_4\in\mathbb{R}\right\}$$



12) A certain corporation has a vector $\vec{b}$ they are unwilling to share. But they have announced that $\vec{x} = \begin{bmatrix} 3 \\ 6.2 \\ 17 \\ 9 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{b}$. Find five more solutions to $A\vec{x} = \vec{b}$. (8 points)

We know that every solution to $A\vec{x} = \vec{b}$ is in the form $\vec{x}_p + \vec{x}_h$, so we need to add homogeneous solutions to our particular solution. Anything obtained as below is a solution.

$$\begin{bmatrix} 3 \\ 6.2 \\ 17 \\ 9 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} x_4$$