Name ___Solutions____ Linear Algebra, Test 2, 11/3/2014

1) Find an equation whose solutions are the eigenvalues for the matrix below. You do not need to fully simplify the equation, but you shouldn't need any linear algebra tools to understand what the equation says. (10 points)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 10 & -4 & 2 \end{bmatrix}$$

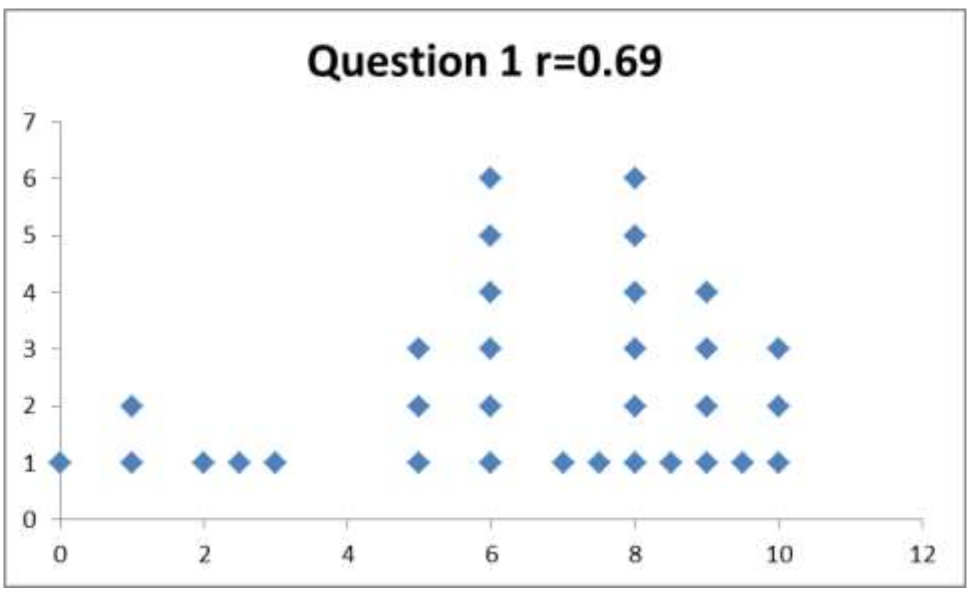
We want to find the determinant of $A - \lambda I_4$, and set that equal to zero:

$$\begin{vmatrix} 0-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 0 & 2 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 10 & -4 & 2-\lambda \end{vmatrix} = 0$$

Evaluating the determinant we get:

$$-\lambda \cdot \left[(5-\lambda)\big((2-\lambda)(2-\lambda)+4\big) + 2 \cdot 10 \cdot (2-\lambda)\right] = 0$$

The most common mistake was missing terms in the determinant, as if this was a diagonal matrix.

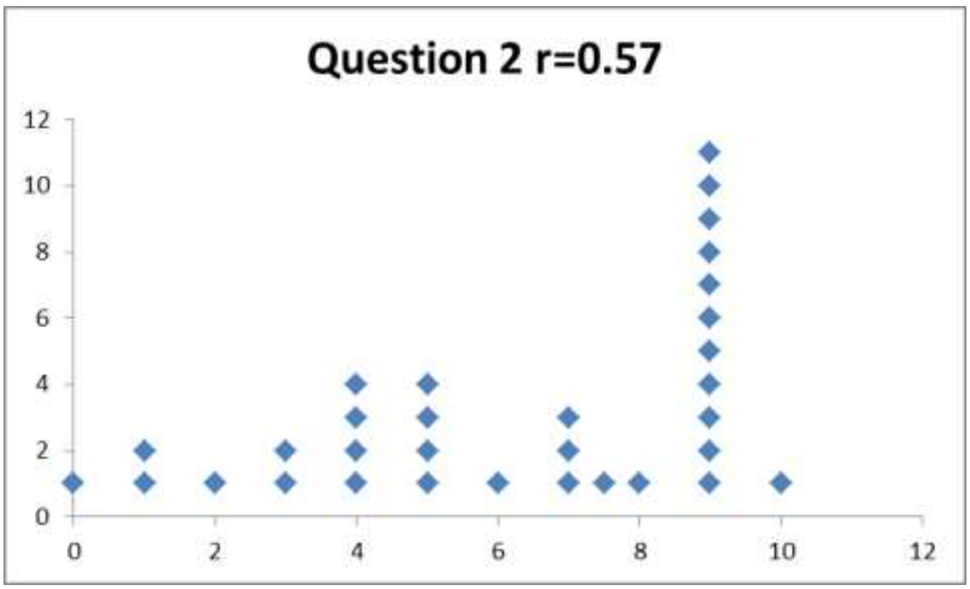


2) A matrix A has two identical columns. A linear transformation T is defined by $T(\vec{x}) = A\vec{x}$. What can you say about the number of solutions to the equation $T(\vec{x}) = \vec{b}$? (10 points)

The identical columns tell us that the columns are linearly dependent. In particular, the equation $T(\vec{x}) = \vec{b}$ has a free variable, so if there are solutions there are infinitely many solutions. There could, however, be no solutions.

We cannot be any more precise because we don't know how much dependence there is: the solution set has dimension 1, or 2, 247, or any positive integer.

Quite a few people realized that there would be infinitely many solutions, but only one of those people noticed that the system itself could be inconsistent.



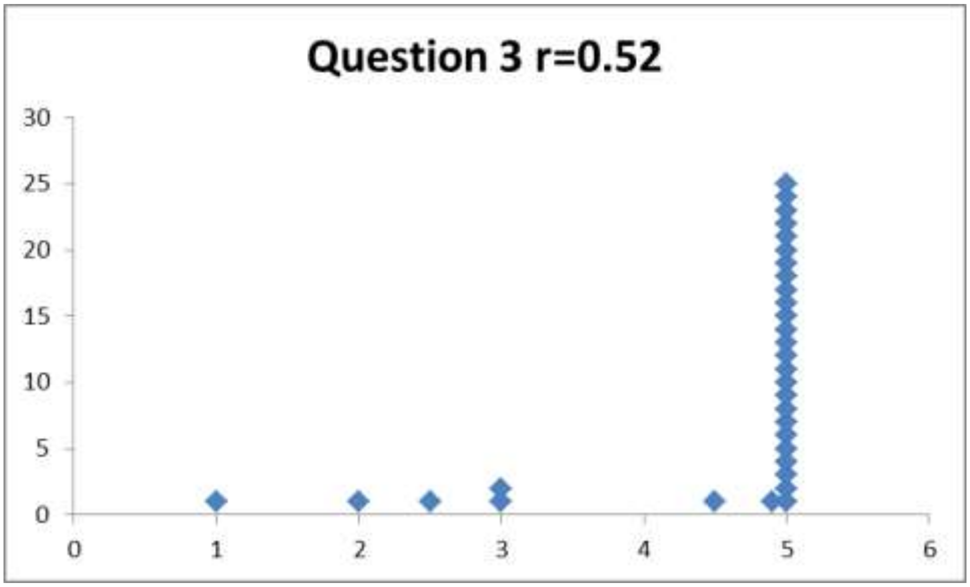
Use the bases below for all the questions on this page. You do not need to simplify any answers at all.

Seriously, don't do any work just write down the answers.

$$B_1 = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\-7\end{bmatrix}\right\} \quad B_2 = \left\{\begin{bmatrix}5\\4\end{bmatrix}, \begin{bmatrix}9\\8\end{bmatrix}\right\} \quad S = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

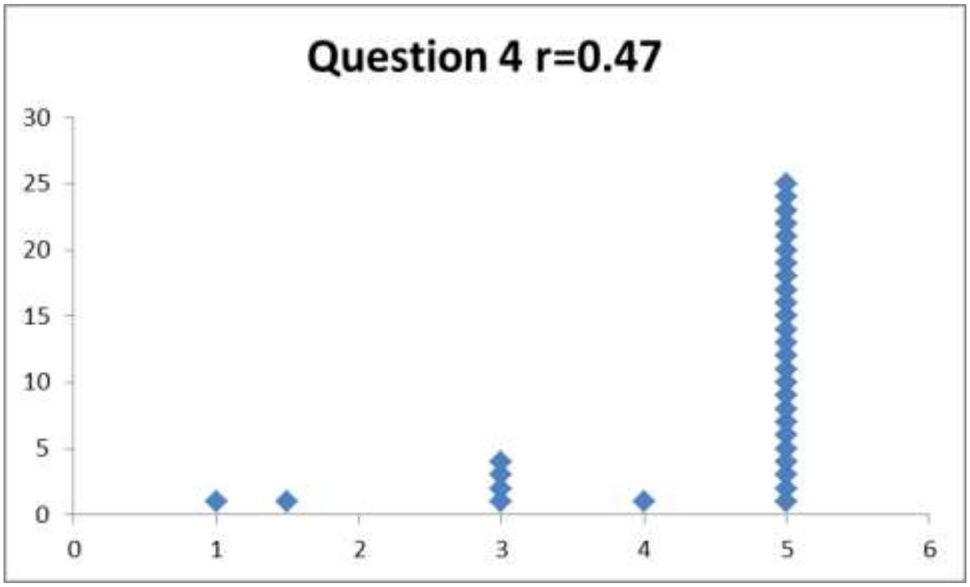
3) What is $\begin{bmatrix}11\\12\end{bmatrix}_{B_1}$ in S coordinates? (5 points)

$$\begin{bmatrix}1 & 3\\2 & -7\end{bmatrix}\begin{bmatrix}11\\12\end{bmatrix}_{B_1}$$



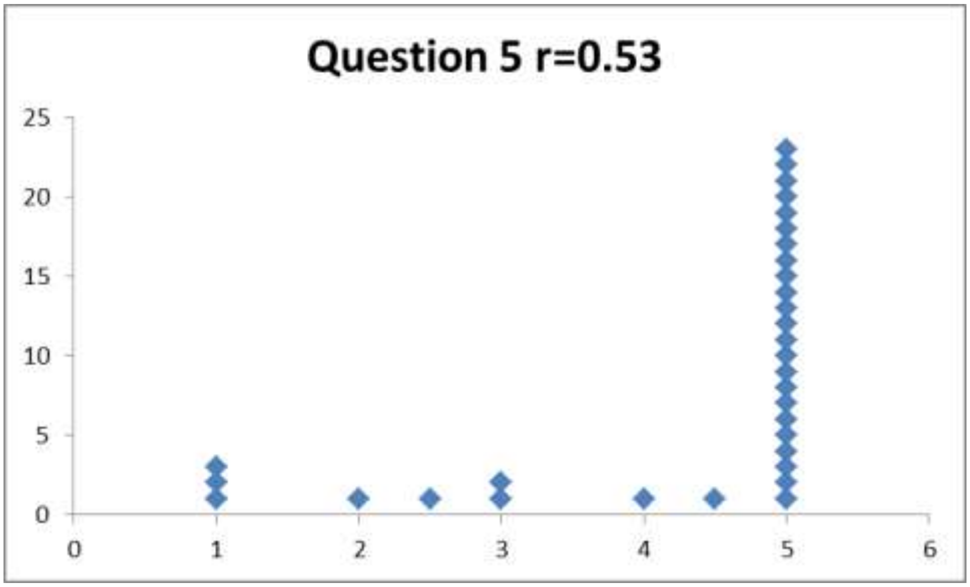
4) What is $\begin{bmatrix} 13 \\ 14 \end{bmatrix}_S$ in B_2 coordinates? (5 points)

$$\begin{bmatrix} 5 & 9 \\ 4 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 13 \\ 14 \end{bmatrix}_S$$



5) What is $\begin{bmatrix}15\\16\end{bmatrix}_{B_1}$ in B_2 coordinates? (5 points)

$$\begin{bmatrix}5 & 9\\4 & 8\end{bmatrix}^{-1}\begin{bmatrix}1 & 3\\2 & -7\end{bmatrix}\begin{bmatrix}15\\16\end{bmatrix}_{B_1}$$



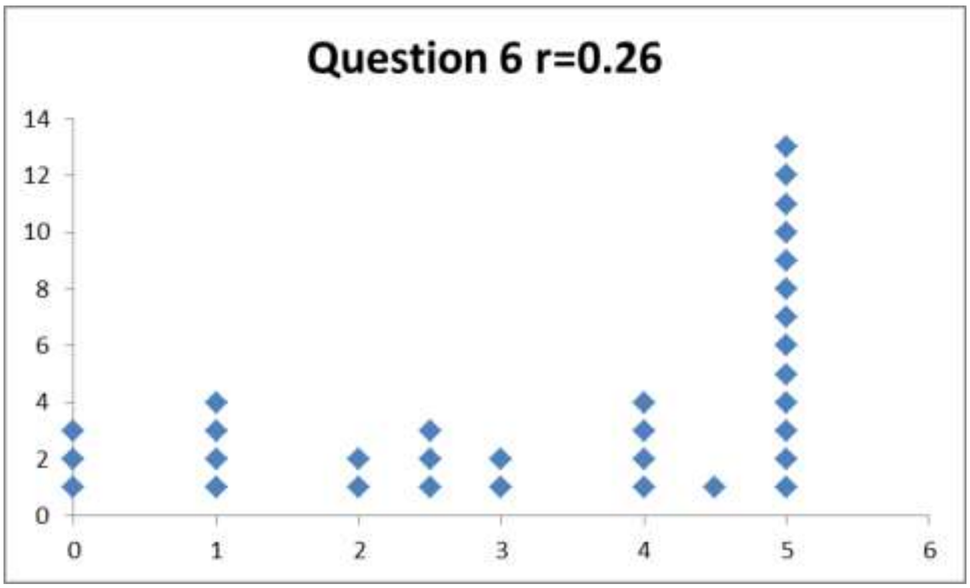
6) The linear operator T does two things simultaneously: (5 points)

- Transforms vectors from S coordinates to B_1 coordinates
- Stretches the vector by a factor of 2 in every direction.

What is the matrix associated to T?

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -7 \end{bmatrix}^{-1} = 2 \begin{bmatrix} 1 & 3 \\ 2 & -7 \end{bmatrix}^{-1}$$

The two scales it in every direction. Think about the matrix $2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$: in the plane this makes the object twice as large in every direction.



7) The linear operator T does three things simultaneously: (3 bonus points)

- Transforms vectors from S coordinates to B_1 coordinates
- Stretches the vector by a factor of 2 in the direction $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Stretches the vector by a factor of 3 in the direction $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

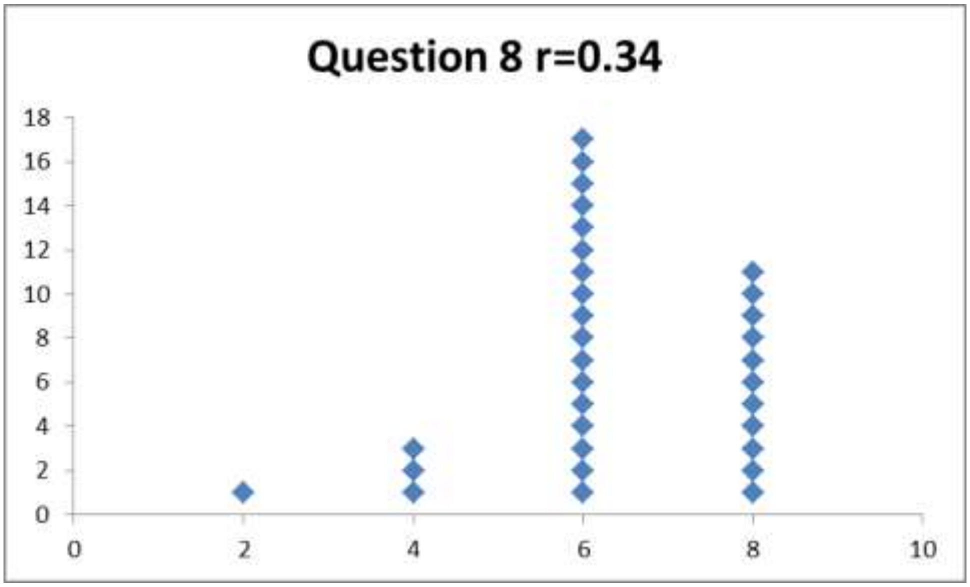
What is the matrix associated to T?

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that the order is important because the 2 scales the 1st direction in the basis S while the 3 scales the 1st direction in the basis B_1.

8) Which of the matrices below are invertible? Circle them. (2 points each)

$$\begin{bmatrix} 17 & 23 \\ 234 & 16 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 20 & 30 \\ 0 & 16 & -23 & 4 \\ 0 & 0 & 13 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & -7 \\ 0 & 1 & 0 & 6 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 4 & 3 & 0 & 1 & 18 \\ -2 & 0 & 0 & 0 & 1 \end{bmatrix}$$



9) Find the inverse of the matrix below. (12 points)

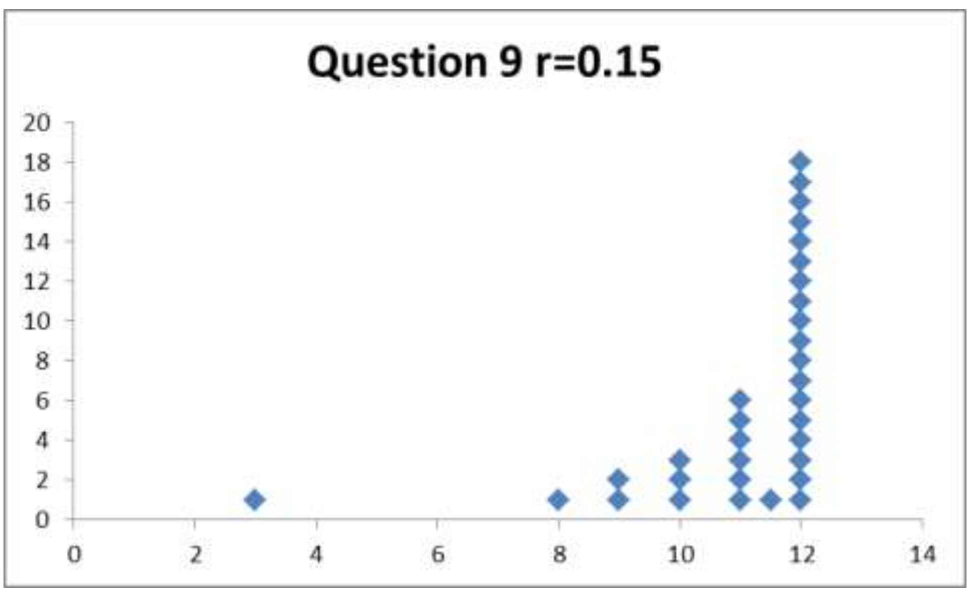
$$\begin{bmatrix} 3 & 6 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 3 & : & 1 & 0 & 0 \\ 1 & 3 & 1 & : & 0 & 1 & 0 \\ 0 & 0 & 2 & : & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & : & \frac{1}{3} & 0 & 0 \\ 1 & 3 & 1 & : & 0 & 1 & 0 \\ 0 & 0 & 2 & : & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & : & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & : & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 1 & -2 & 0 \\ 0 & 1 & 0 & : & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 2 & : & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & : & 1 & -2 & 0 \\ 0 & 1 & 0 & : & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & : & 0 & 0 & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 & -2 & -\frac{1}{2} \\ 0 & 1 & 0 & : & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & : & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Hence the inverse is:

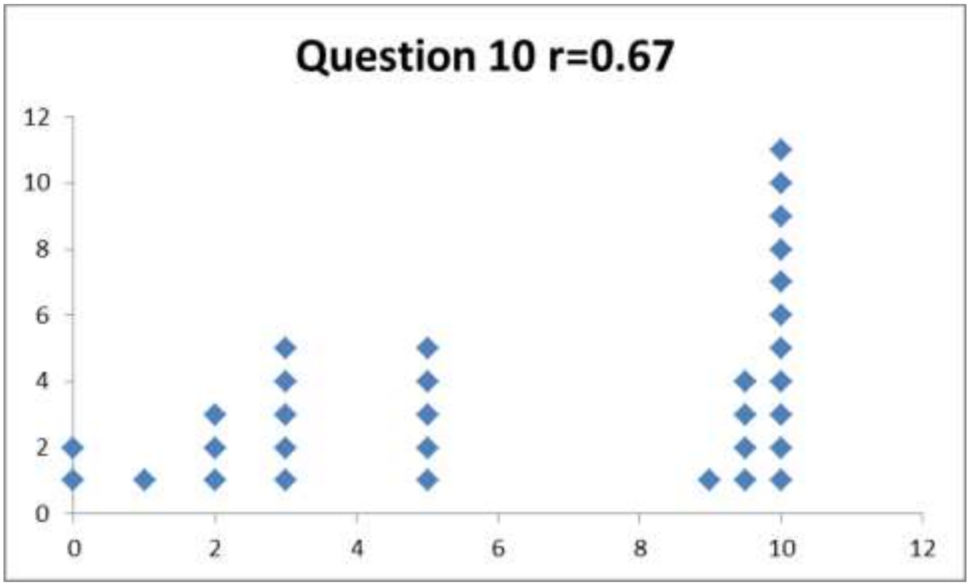
$$\begin{bmatrix} 1 & -2 & -\frac{1}{2} \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$



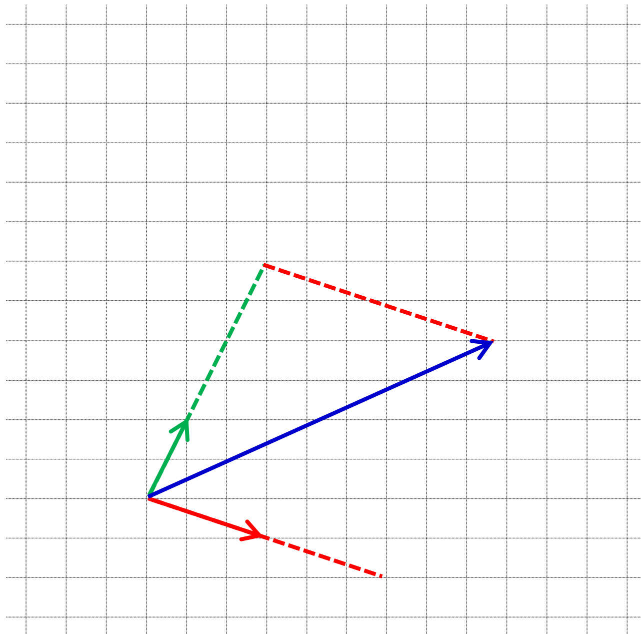
10) A 5×5 matrix has determinant 3. What can you say about its null space? (10 points)

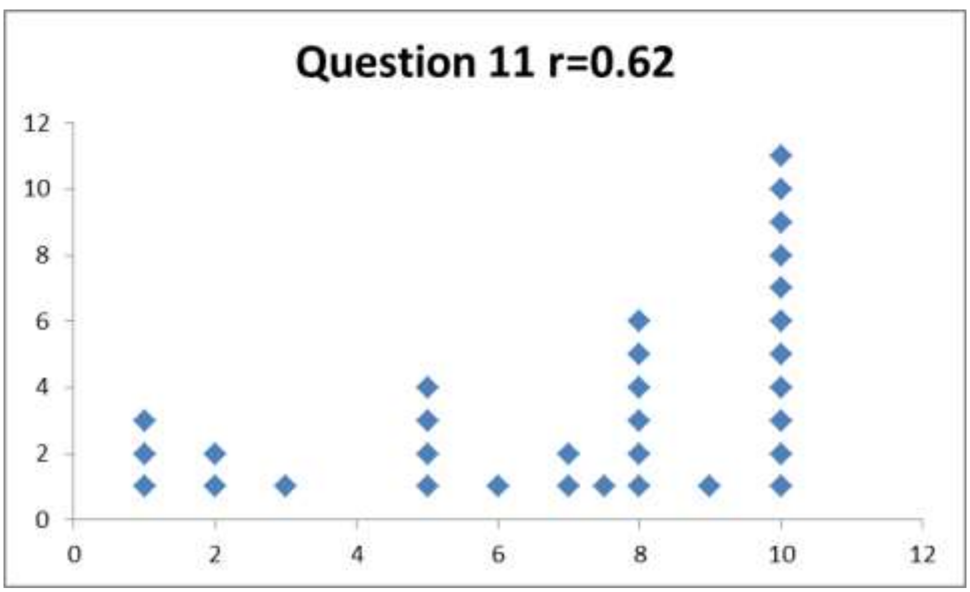
The null space is $\{\vec{0}\}$. We know that it is trivial because the matrix is invertible: its determinant isn't 0.

This question ended up being bimodal I think because some people were in the correct ballpark and some were not.



11) Using the basis $B = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\end{bmatrix}\right\}$, graphically illustrate the sum $\begin{bmatrix}3\\0\end{bmatrix}_B + \begin{bmatrix}0\\2\end{bmatrix}_B$. (10 points)





12) Let $T: \mathbb{R}^6 \to \mathbb{R}^7$ and $S: \mathbb{R}^7 \to \mathbb{R}^{12}$ be two linear operators. It is known that the equation $S\big(T(\vec{x})\big) = \vec{b}$ can be solved uniquely, but another equation $S(\vec{x}) = \vec{b}_2$ has multiple solutions. Expand on this and explain as precisely as possible exactly how many solutions the equation $S(\vec{x}) = \vec{b}_2$ has. (3 bonus points)

Short answer:

The dimension of the solution set to $S(\vec{x}) = \vec{b}_2$ is 1. We obtain this by applying the rank nullity theorem:

Because $S\big(T(\vec{x})\big) = \vec{b}$ can be solved uniquely, the rank of both S and T must be 6.

Because $S(\vec{x}) = \vec{b}_2$ has multiple solutions, the rank of S must be at most 6.

Hence the rank of S is 6, so the nullity of S is 1. Hence the dimension of the solution set to $S(\vec{x}) = \vec{b}_2$ is 1.

Longer answer:

From the fact that $S\big(T(\vec{x})\big) = \vec{b}$ has a unique solution, we know that T must have rank 6, and that S must have rank at least 6.

(If T had rank less than 6, there would be infinitely many solutions because T would not be one to one, so $S \circ T$ is also not one-to-one)

(Getting our hands on S is a little trickier, because $S\big(T(\cdot)\big)$ only allows the range of T to be applied to S. So let's look at each possible rank of S:

- If S has rank 7, then $S(\vec{x}) = \vec{b}_2$ could not have multiple solutions, so that is out.
- The rank of S could be 6.
- If S has rank 5, then the range of S has dimension 5. However, the range of $S \circ T$ has dimension 6. This is impossible because the range of $S \circ T$ is a subset of the range of S.
- Similarly every rank less than 4 is impossible

As we see, the only possible rank for S is 6. Essentially what's happening is that T maps into just the messy part of the domain of S and misses the kernel of S.

Because the rank of S is 6, we know that $S(\vec{x}) = \vec{b}_2$ has either zero solutions, or a one dimensional solution set. We know that it has solutions, so we know it has exactly one dimension of solutions.

13) Find all the eigenvectors for the matrix below. You may include $\vec{0}$ even though we don't consider it an eigenvector if it makes your answer(s) easier to write down. (10 points)

$$\begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$$

First we find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda)+3 = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4) = 0$$

Therefore we have two eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = 4$.

Next we find the eigenvectors.

For the eigenvalue $\lambda_1 = 2$:

$$\begin{bmatrix} 1-2 & 3 \\ -1 & 5-2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}$$

We see that the null space is given by the solutions to $-x_1 + 3x_2 = 0$, hence the set of eigenvectors here is:

$$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} x_2 : x_2 \in \mathbb{R} \right\}$$

For the eigenvalue $\lambda_1 = 4$:

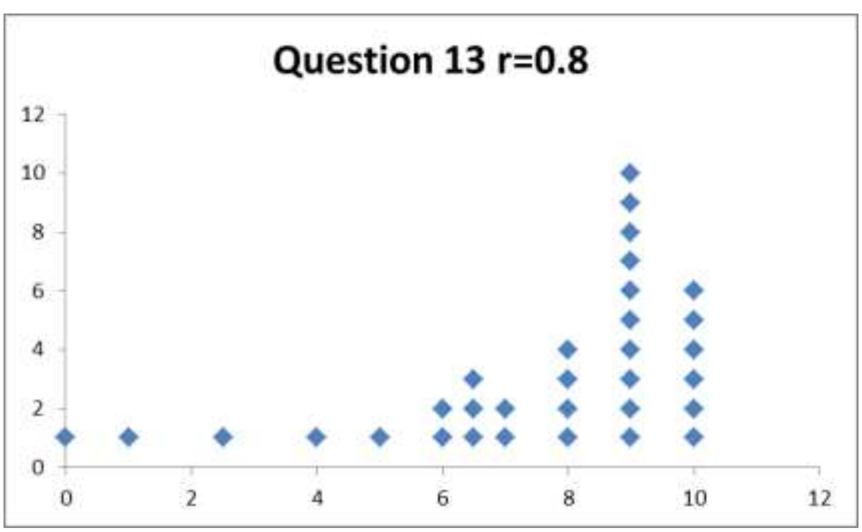
$$\begin{bmatrix} 1-4 & 3 \\ -1 & 5-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

We see that the null space is given by the solutions to $-x_1 + x_2 = 0$, hence the set of eigenvectors here is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 : x_2 \in \mathbb{R} \right\}$$

Therefore all eigenvectors are:

$$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} x_2 : x_2 \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 : x_2 \in \mathbb{R} \right\}$$

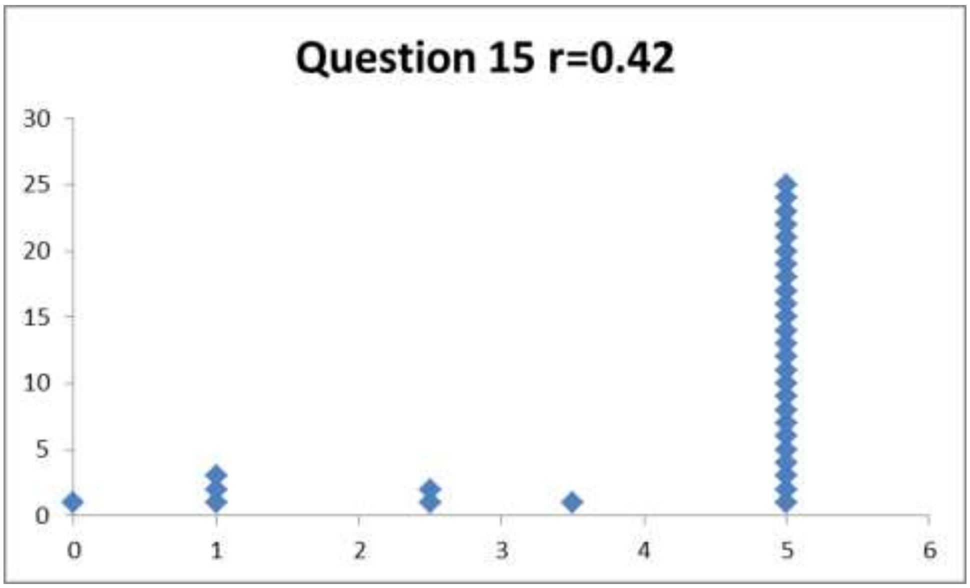


14) How would you explain the number of eigenvectors in the previous problem in terms of dimension? (3 bonus points)

There are two one dimensional sets of eigenvectors. Note that this is not a two dimensional set.

15) Find $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$ (5 points)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot(-1)+2\cdot 1 & 1\cdot 2+2\cdot 1 \\ 3\cdot(-1)+4\cdot 1 & 3\cdot 2+4\cdot 1 \\ 0\cdot(-1)+2\cdot 1 & 0\cdot 2+2\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 10 \\ 2 & 2 \end{bmatrix}$$



16) A 7×7 matrix A has determinant 4. Find the determinant of A^{-1}. (5 points)

$$|A^{-1}| = \frac{1}{|A|} = \frac{1}{4}$$