Name ___Solutions____ Linear Algebra, Test 3, 12/1/2014

1) Diagonalize the matrix below. (20 points)

$$\begin{bmatrix} 6 & 4 \\ 2 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 6-x & 4 \\ 2 & -1-x \end{vmatrix} = (6-x)(-1-x) - 2\cdot 8 = x^2 - 5x - 14 = (x-7)(x+2)$$

Hence we have two eigenvalues: $\lambda_1 = 7, \lambda_2 = -2$.

Now let's find the eigenvectors:

$\begin{bmatrix} 6-7 & 4 \\ 2 & -1-7 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$. Hence we need to solve the equation $x_1 = -4x_2$. We get $\vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

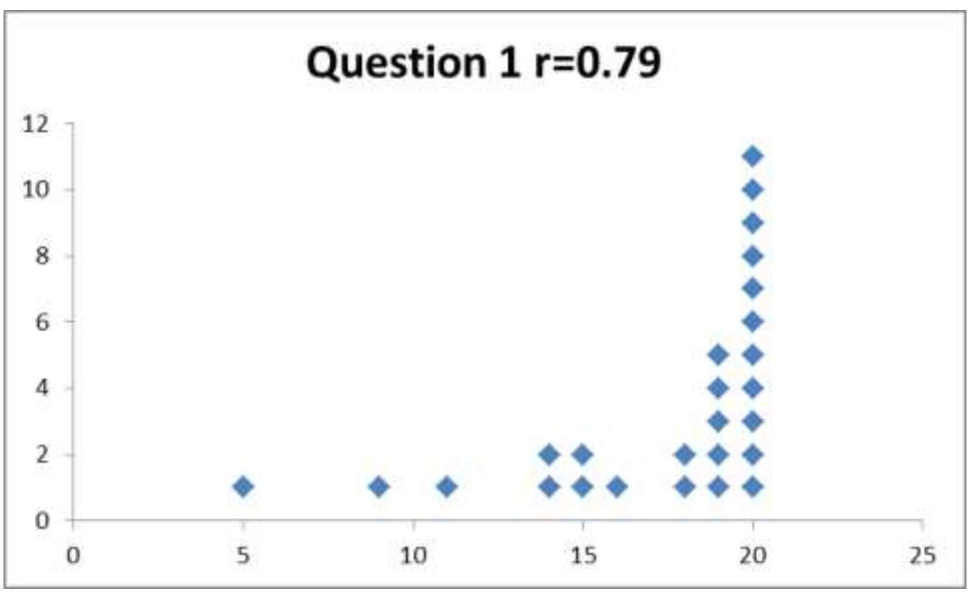
$\begin{bmatrix} 6+2 & 4 \\ 2 & -1+2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$. Hence we need to solve the equation $2x_1 = x_2$. We get $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Putting this together we find that the diagonalization is:

$$\begin{bmatrix} 6 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 6 & 4 \\ 2 & -1 \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ -1 & 4 \end{bmatrix}$$

Note that there are other possible answers depending on the order of the eigenvalues and which scalar multiple of each eigenvector you chose.



2) Alice is wandering down a red concrete road, where she happens upon the following vectors:

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}3\\1\\2\\0\end{bmatrix}, \text{and then} \begin{bmatrix}9\\2\\2\\1\end{bmatrix}$$

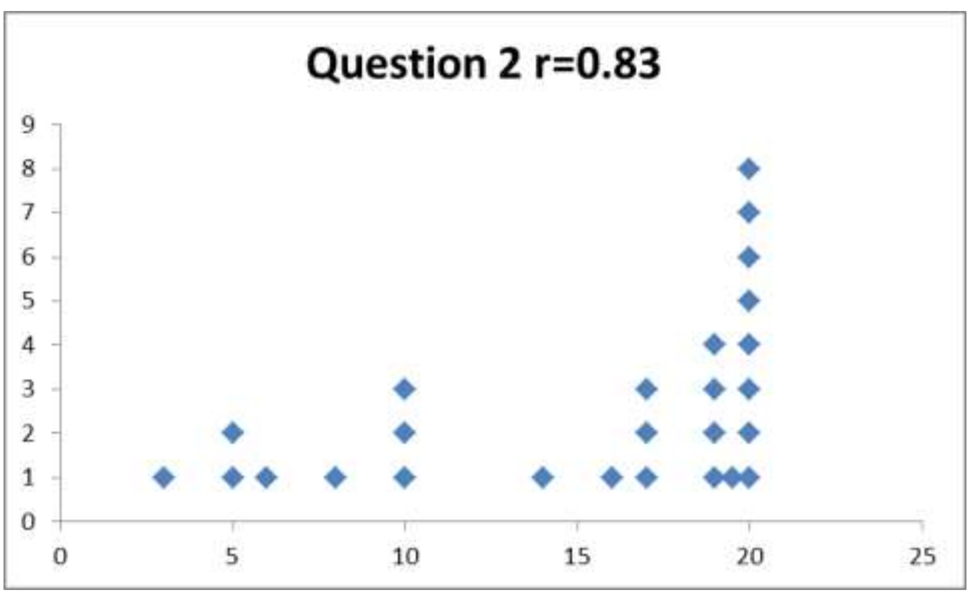
"Ah-ha!" She exclaimed, "These vectors are linearly independent." She continued down the road until she reached the gates of Sapphire Suburbia. Much to her chagrin, Alice discovered that they would not let her in if she carried vectors that were not orthogonal to each other. Can we aid Alice by orthogonalizing her vectors? Yes! Find three mutually orthogonal vectors that encode the same information as Alice's three vectors. (20 points)

After reading the question, we see that we are tasked with orthogonalizing the three given vectors. This is the Gram-Schmidt process.

$$\vec{v}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

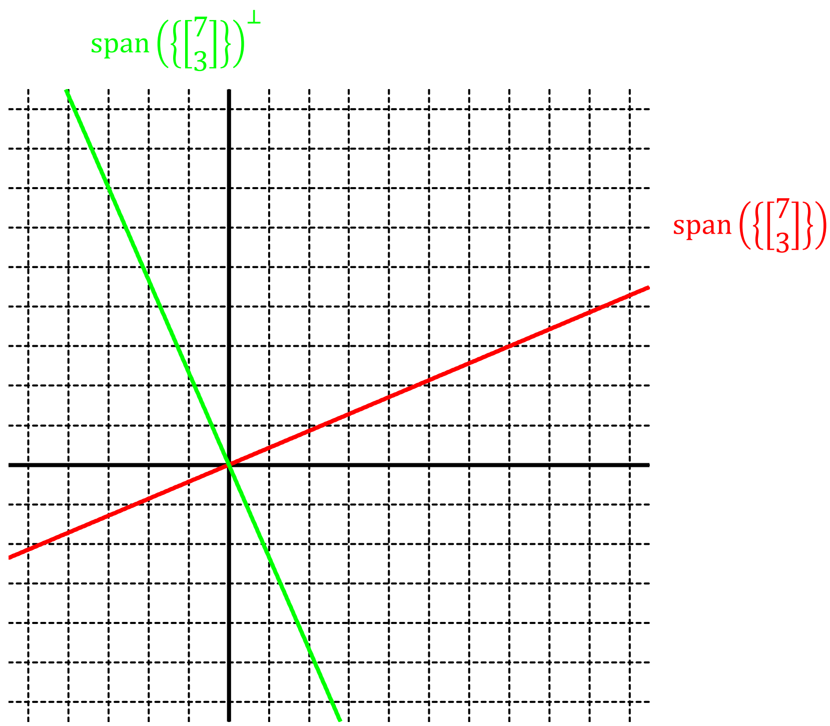
$$\vec{v}_2 = \begin{bmatrix}3\\1\\2\\0\end{bmatrix} - \frac{3}{1}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\2\\0\end{bmatrix}$$

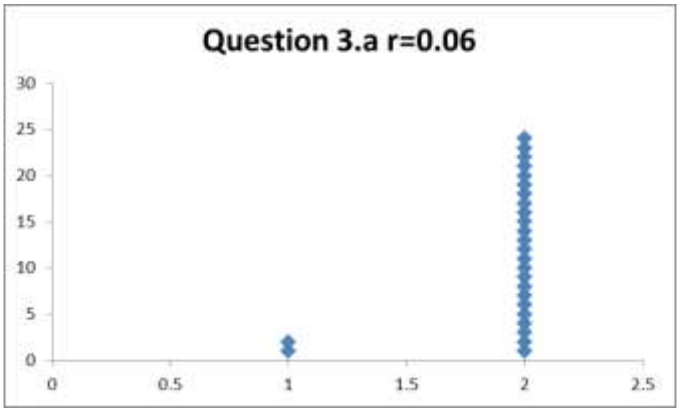
$$\vec{v}_3 = \begin{bmatrix}9\\2\\2\\1\end{bmatrix} - \frac{9}{1}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \frac{2+4}{1+4}\begin{bmatrix}0\\1\\2\\0\end{bmatrix} = \begin{bmatrix}0\\2\\2\\1\end{bmatrix} - \begin{bmatrix}0\\ {}^{6}/_{5}\\ {}^{12}/_{5}\\ 0\end{bmatrix} = \begin{bmatrix}0\\ {}^{4}/_{5}\\ -{}^{2}/_{5}\\ 1\end{bmatrix}$$

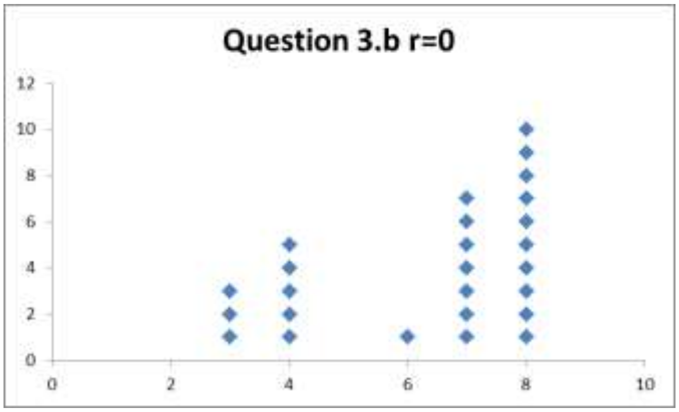


3) On the space provided, illustrate both of the following:

- The space $\text{span}\left(\left\{\begin{bmatrix}7\\3\end{bmatrix}\right\}\right)$ (2 points)
- The orthogonal complement of the space in the previous bullet point. (8 points)







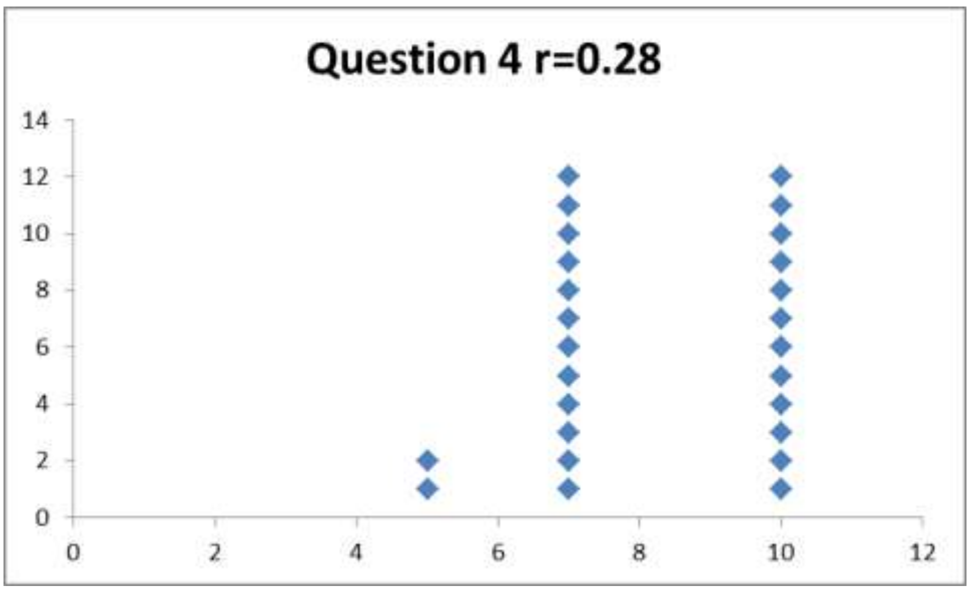
4) Consider the diagonalization given below and the basis $B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. Note the order on the basis.

$$\begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$$

Find $\begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}^{17}$. (10 points. As long as your answer clearly illustrates the main point of this question, you do not need to simplify it.)

$$\begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}^{17} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix}^{17} \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$$

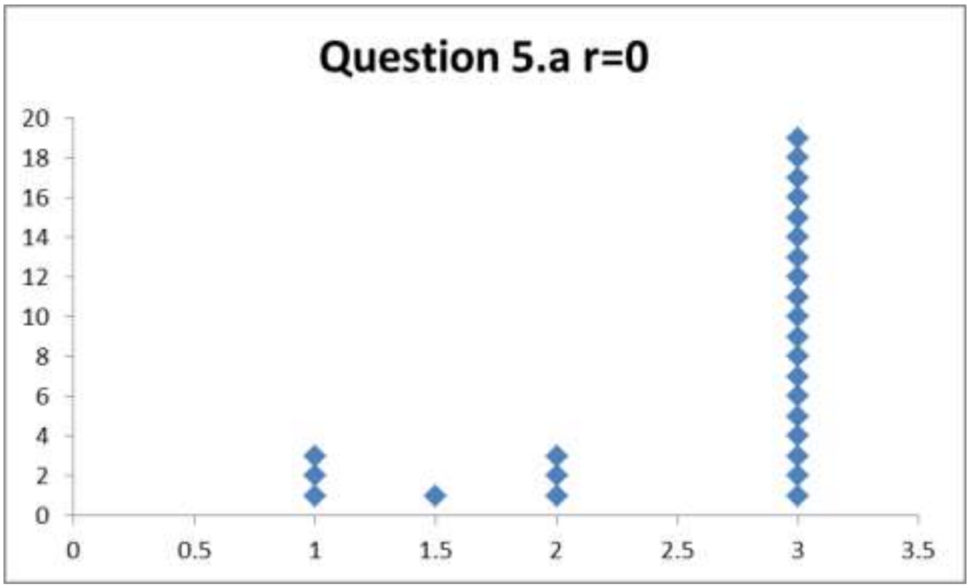
$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 9^{17} & 0 \\ 0 & 3^{17} \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$$



5) Let A be a 2×2 matrix with characteristic polynomial $ax^2 + bx + c$. Also, A has two distinct eigenvalues λ_1 and λ_2. Use this information to answer the following questions.

5.a) What could a be? (3 points)

1 or -1. (Actually it's always 1 because that's how the characteristic polynomial is defined. But I would have accepted -1 as well)

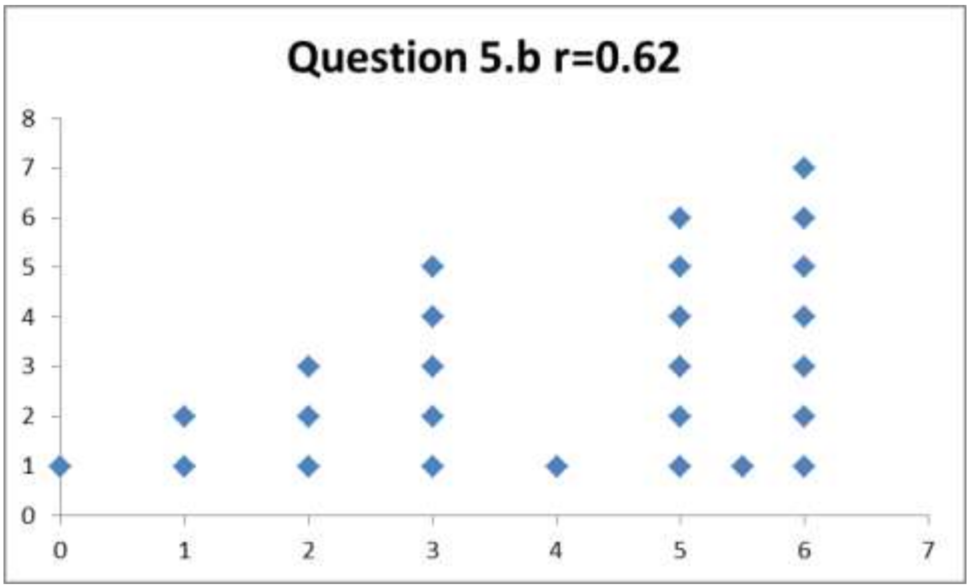


5.b) What is b? (6 points)

Note that $ax^2 + bx + c = (x - \lambda_1)(x - \lambda_2) = x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2$.

Hence $b = -\lambda_1 - \lambda_2$

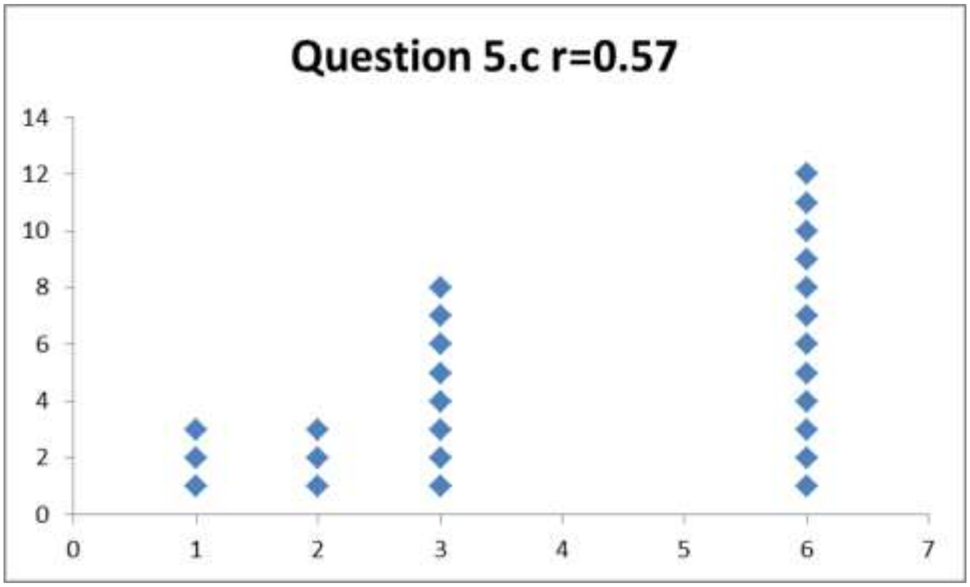
(Or the negation of that if a was -1)



5.c) What is c? (6 points)

From the work for the previous question, we see that $c = \lambda_1 \lambda_2$.

(Or the negation of that if a was -1)

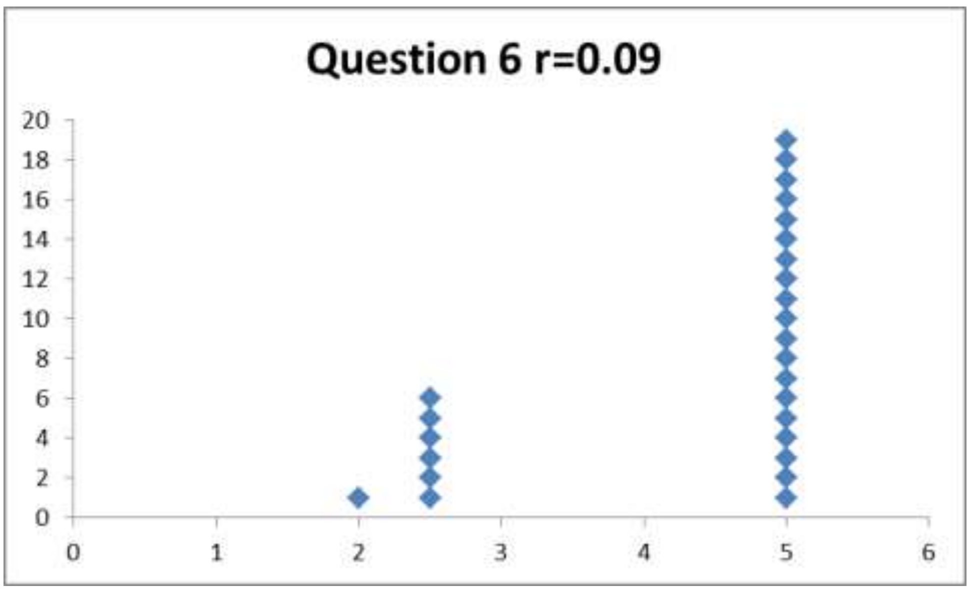


6) If $A \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, what is $A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$? (5 points)

$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is an eigenvector with eigenvalue 3, so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is also an eigenvector with eigenvalue 3. Hence $A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

Or you could factor out and cancel the two:

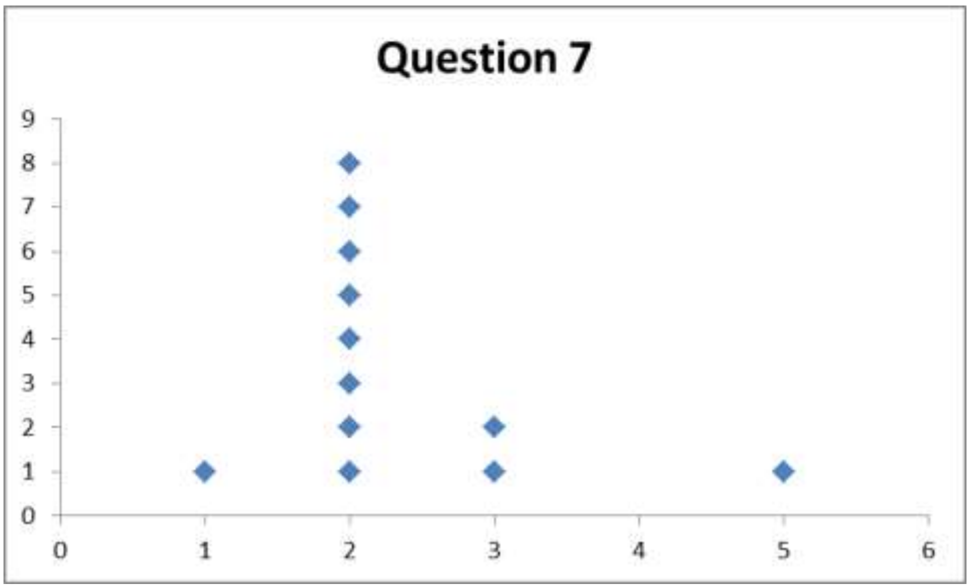
$$A \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$\therefore 2A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \cdot 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\therefore A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$



7) If $A\vec{x} = 3\vec{x}$, is it completely necessary that $\vec{x}$ is an eigenvector? Why or why not? (5 bonus points)

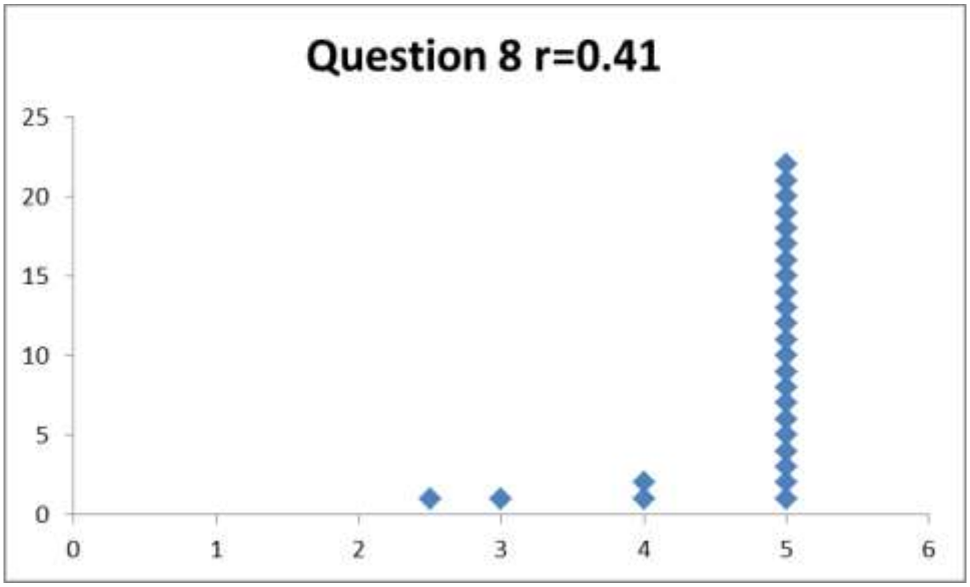
Usually, yes, but not always. This equation is the definition of an eigenvector: An eigenvector is any nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{C}$.

But…nonzero. No matter what A is, this equation is always true for the zero vector, which is not considered an eigenvector.



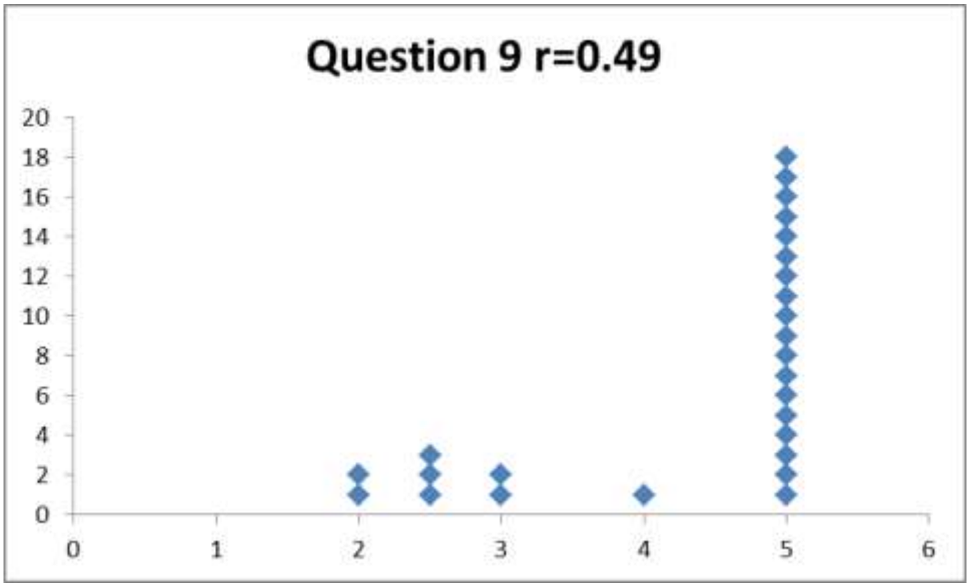
8) Find the length of the vector $\begin{bmatrix}2\\3\\7\end{bmatrix}$. (5 points)

$$\left\|\begin{bmatrix}2\\3\\7\end{bmatrix}\right\| = \sqrt{2^2+3^2+7^2} = \sqrt{62}$$



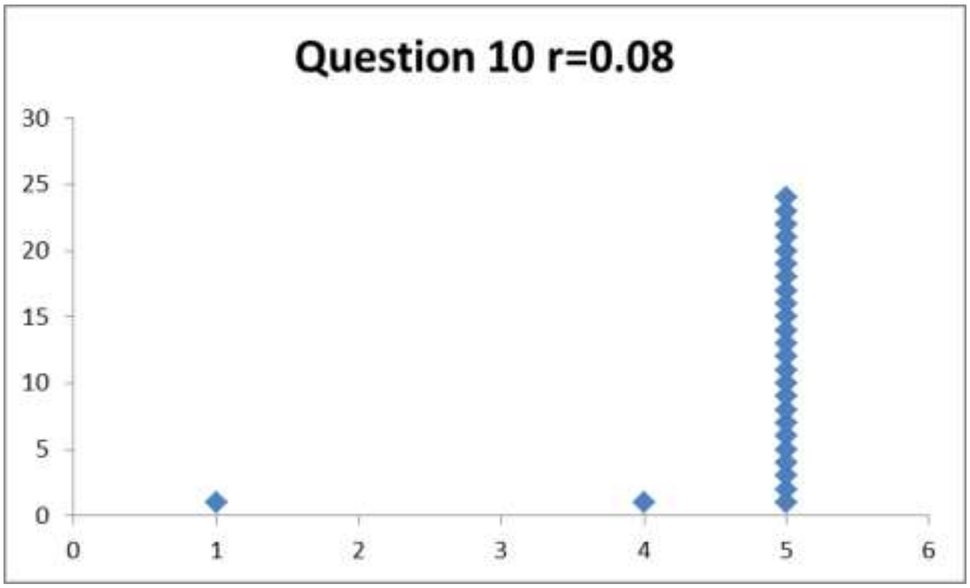
9) Find the distance between the two vectors $\begin{bmatrix}2\\7\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$. (5 points)

$$\left\|\begin{bmatrix}2\\7\end{bmatrix}-\begin{bmatrix}3\\4\end{bmatrix}\right\| = \left\|\begin{bmatrix}-1\\3\end{bmatrix}\right\| = \sqrt{1^2+3^2} = \sqrt{10}$$



10) Find $\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}\bullet\begin{bmatrix}0\\2\\-1\\0\\1\\0.5\end{bmatrix}$. (5 points)

$$2\cdot 2+3\cdot(-1)+5\cdot 1+6\cdot 0.5=4-3+5+3=9$$



11) On the palette below, illustrate the norm of the vector $\begin{bmatrix}3\\4\end{bmatrix}$. (5 points)