Name ____Solutions____ Linear Algebra, Test 2

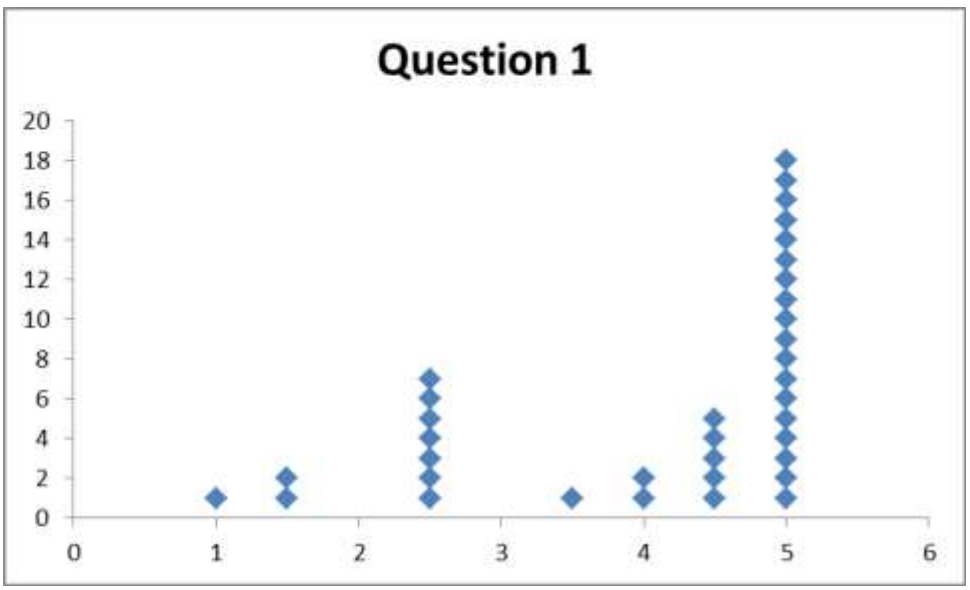
Throughout the test simplify all answers except where stated otherwise. For questions in which the answer is a single number, word, etc, show work or provide an explanation for partial credit; otherwise no credit will be given for incorrect responses.

1) Determine whether or not an inverse exists for the matrix shown below. (5 points)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 2 \\ 3 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The inverse does not exist: we can see this because the columns are linearly dependent: $C_1 + C_3 = C_4$

(Most people calculated the determinant and found that it was zero. This also works but took more work.)



2) Find the inverse of the matrix shown below. (15 points)

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 7 & 8 \end{bmatrix}$$

We will reduce the matrix to the identity while applying the same elementary row operations to the identity.

$$\begin{bmatrix} 1 & 2 & 0 & : & 1 & 0 & 0 \\ 1 & 3 & 0 & : & 0 & 1 & 0 \\ 0 & 7 & 8 & : & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 0 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 0 \\ 0 & 7 & 8 & : & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 7R_2$

$R_1 \to R_1 - 2R_2$

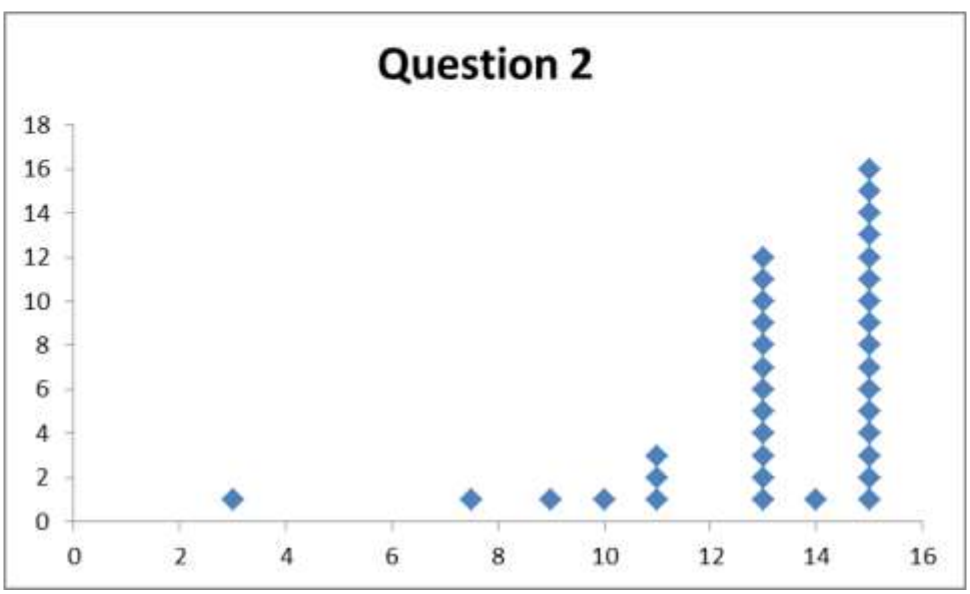
$$\begin{bmatrix} 1 & 0 & 0 & : & 3 & -2 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 0 \\ 0 & 0 & 8 & : & 7 & -7 & 1 \end{bmatrix}$$

$R_3 \to \frac{1}{8} R_3$

$$\begin{bmatrix} 1 & 0 & 0 & : & 3 & -2 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 0 \\ 0 & 0 & 1 & : & {}^{7}/_{8} & -{}^{7}/_{8} & {}^{1}/_{8} \end{bmatrix}$$

Hence we see that the inverse is:

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 7 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ \frac{7}{8} & -\frac{7}{8} & \frac{1}{8} \end{bmatrix}$$



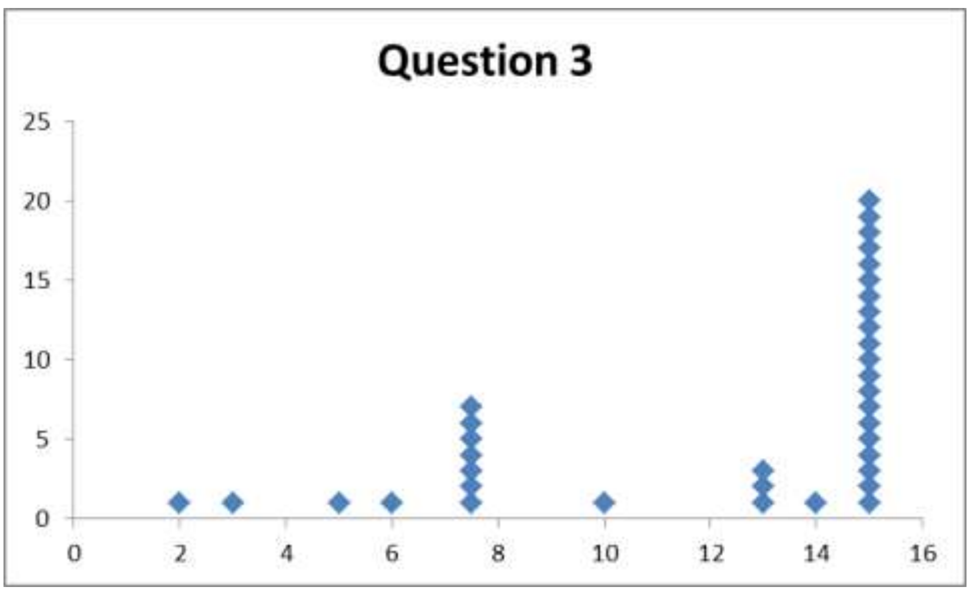
$$B_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} \right\}, B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} \right\},$$

3) Using the bases above, write the vector below in terms of basis B_2. You do not need to simplify your answer. (15 points)

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{B_1}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 7 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 6 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{B_1}$$

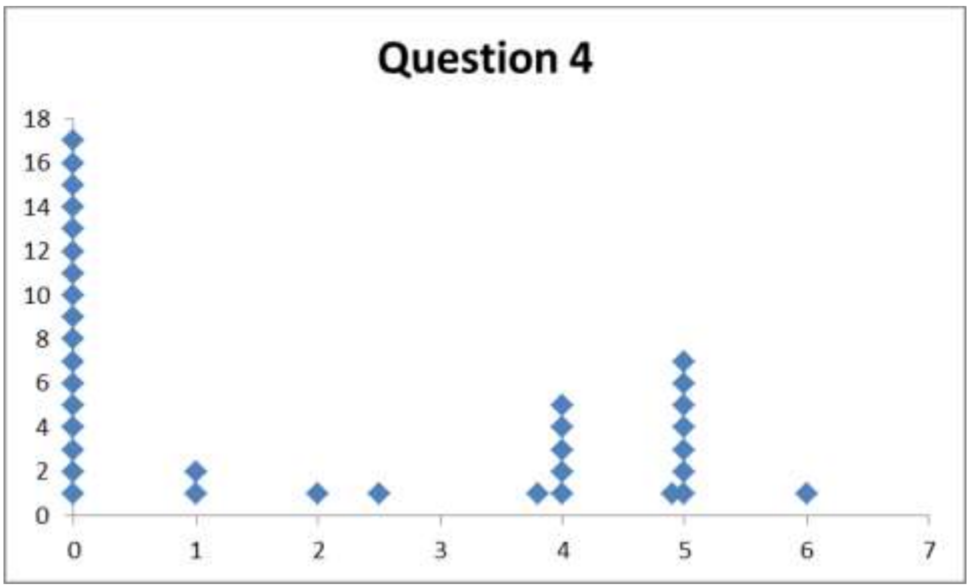
This happens to equal $\begin{bmatrix} -40 \\ 21 \\ -{}^{145}/_{8} \end{bmatrix}_{B_2}$, but it was not necessary to simplify.



4) A 10×10 matrix has matrix has the following eigenvalues. What can you say about the number of linearly independent eigenvectors? (5 points)

$$\lambda_1 = 2 \text{ with multiplicity 4.}$$
$$\lambda_2 = 1 \text{ with multiplicity 2.}$$
$$\lambda_3 = 0 \text{ with multiplicity 1.}$$
$$\lambda_4 = \sqrt{\pi} \text{ with multiplicity 1.}$$
$$\lambda_5 = e^{\pi} \text{ with multiplicity unknown.}$$

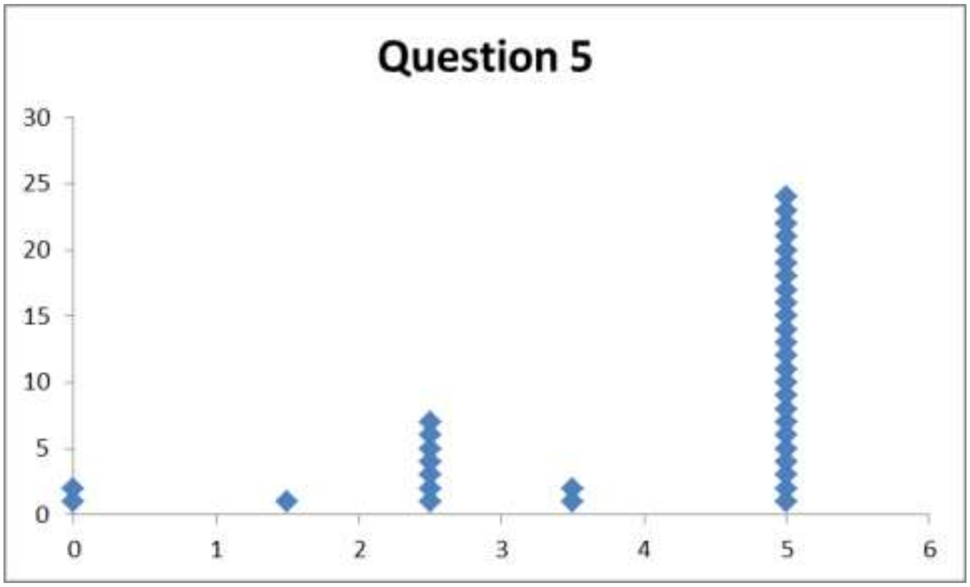
There are at least 5 linearly independent eigenvectors, and at most 10 of them. We know this because every eigenvalue has an eigenvector, and eigenvectors corresponding to different eigenvalues are linearly independent. However, when an eigenvalue has multiplicity greater than 1, we do not know a priori how many linearly independent eigenvectors there are.



5) Find the column space of the matrix below. (5 points)

$$\begin{bmatrix} 2 & 3 & 5 & 7 \\ 1 & 3 & 4 & 9 \end{bmatrix}$$

$$\operatorname{span}\left(\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}3\\3\end{bmatrix}\right\}\right) = \operatorname{span}\left(\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}3\\3\end{bmatrix},\begin{bmatrix}5\\4\end{bmatrix},\begin{bmatrix}7\\9\end{bmatrix}\right\}\right)$$



6) $T: \mathbb{R}^{12} \to \mathbb{R}^{17}$ is a linear operator. It is known that there are at least 6 linearly independent vectors in the codomain that are not in the range of T. What can you say about the solution set of the following equation? (15 points)

$$T(\vec{x}) = \vec{0}$$

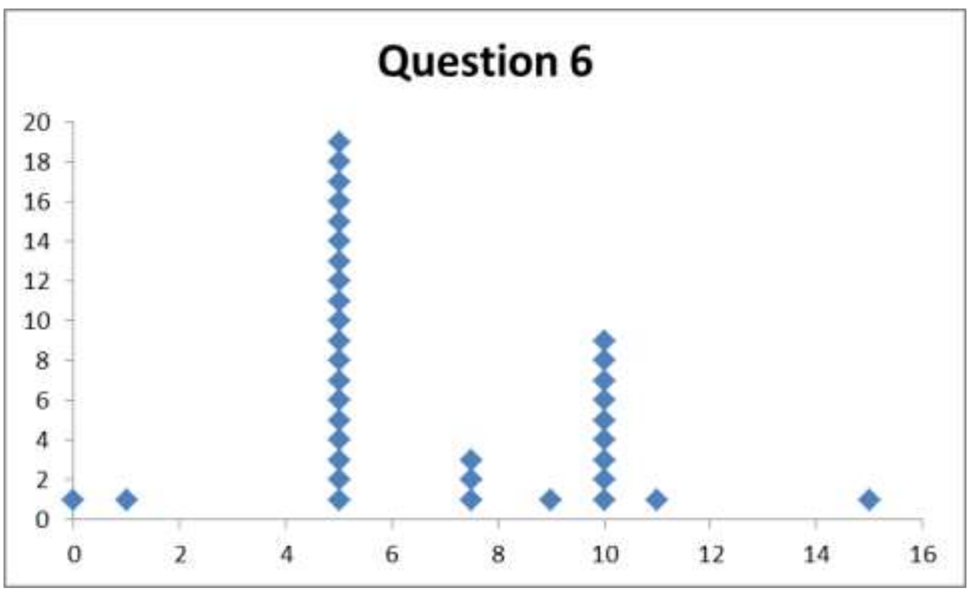
It's nontrivial. That is, we can say that there is some vector $\vec{x}$ other than the zero vector that solves $T(\vec{x}) = \vec{0}$.

To see this we'll use the rank nullity theorem on the linear operator:

$$\dim(\text{range}) + \dim(\text{kernel}) = \dim(\text{domain})$$
$$??+??= 12$$

Now let's look at the dimension of the range. We know that the output of the linear operator lives in $\mathbb{R}^{17}$, so clearly the dimension of the range is at most 17. Looking at the domain, $\mathbb{R}^{12}$, we see that the dimension of the range is at most 12. But we also know that there are 6 linearly independent vectors in the codomain that are not in the range. Hence at least 6 of the 17 dimensions of $\mathbb{R}^{17}$ are outside of the range, meaning that the range has dimension at most $17 - 6 = 11$.

Thus by the rank nullity theorem, the dimension of the kernel is at least 1.



7) Find all the eigenspaces of the matrix $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$. (20 points)

$$c_A(x) = (x+1)[(1-x)(2-x)-6)] = (x+1)(x^2-3x-4) = (x+1)^2(x-4)$$

(Do not expand the cubic polynomial completely: you already have a factor of $+1$!!!)

$\lambda = -1$:

$$\begin{bmatrix} -1+1 & 0 & 0 \\ 0 & 1+1 & 2 \\ 0 & 3 & 2+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Hence the eigenspace for $\lambda = -1$ is:

$$\operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}\right)$$

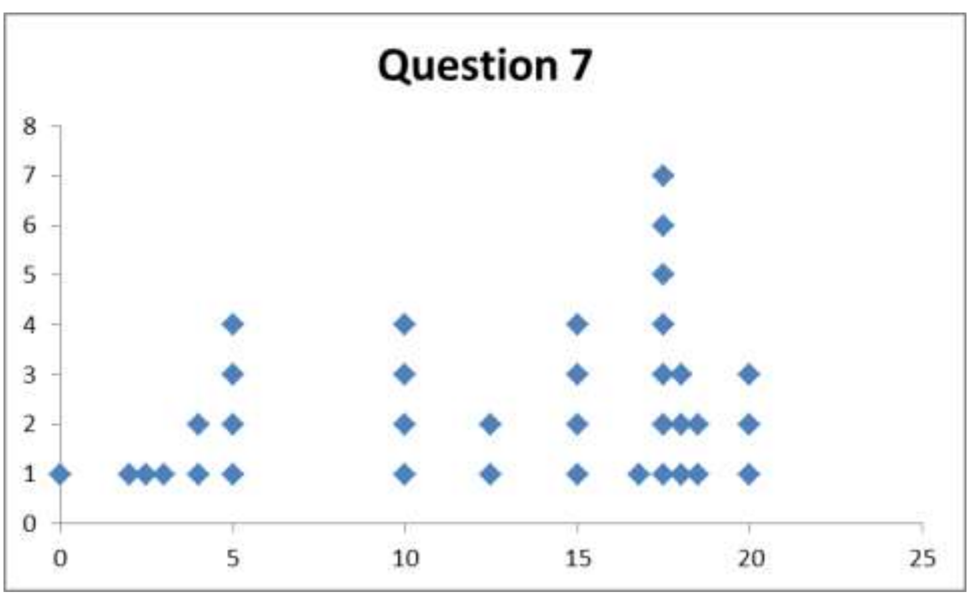
$\lambda = 4$:

$$\begin{bmatrix} -1-4 & 0 & 0 \\ 0 & 1-4 & 2 \\ 0 & 3 & 2-4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

Hence the eigenspace for $\lambda = 4$ is:

$$\operatorname{span}\left(\left\{\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}\right\}\right)$$



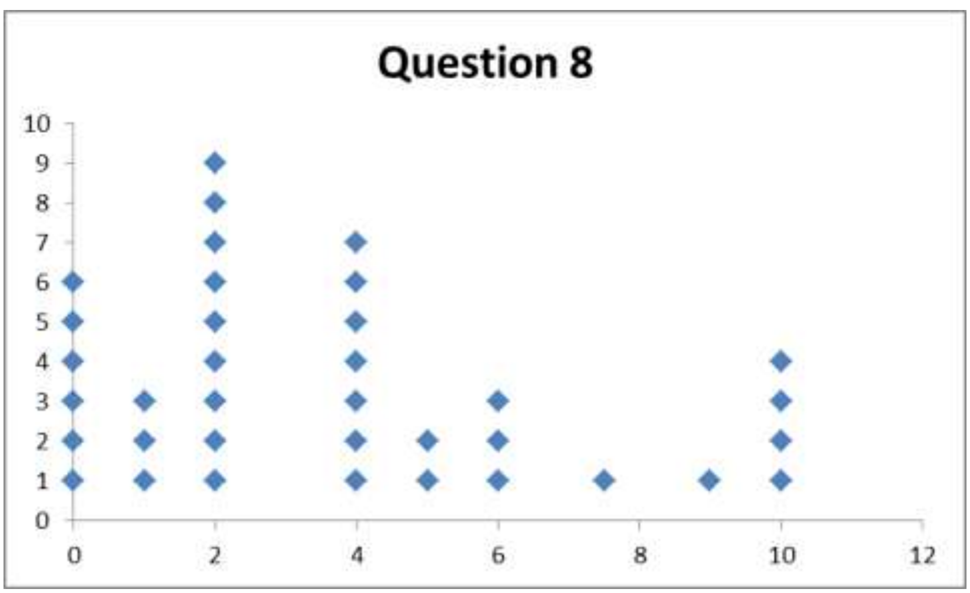
8) A $n \times n$ matrix is known to have an eigenvalue of 0. What else can you say? 2 points for each correct statement, -2 points for each incorrect statement. (10 points maximum)

The matrix is not invertible.
The determinant of the matrix is 0.
The null space of the matrix is nontrivial.
The rank of the matrix is less than n.
The columns of the matrix are linearly dependent.
The rows of the matrix are linearly dependent.
When reduced to echelon form, the matrix has a row of zeroes.
The columns of the matrix do not form a basis.
The rows of the matrix do not form a basis.

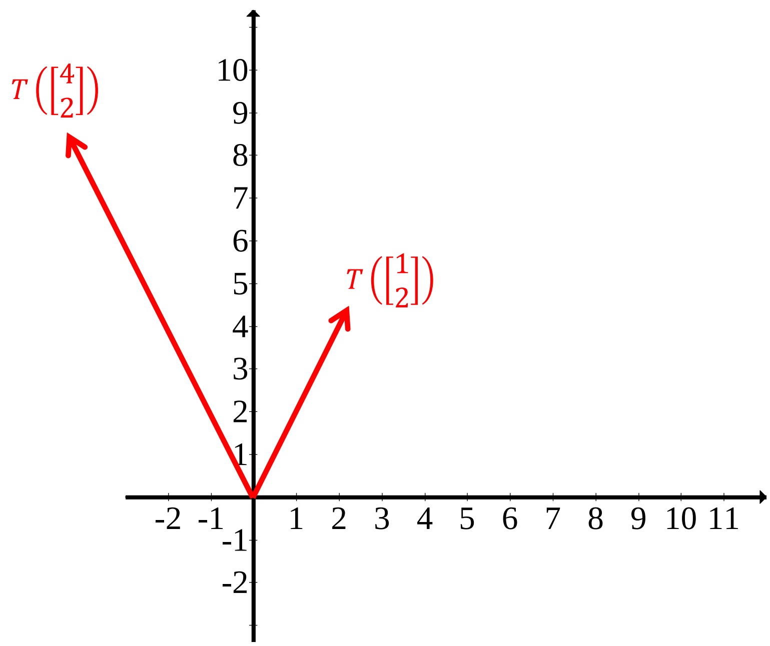
The dimension of the column space is less than n.
The dimension of the row space is less than n.

The associated linear operator is not 1-1.
The associated linear operator is not onto.
The associated linear operator is not invertible.
The kernel of the associated linear operator is nontrivial.

The associated system of homogenous linear equations has a nontrivial solution.
Any associated system of nonhomogenous linear equations that has a solution actually has infinitely many solutions.



9) $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator. It is known that $\begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector of T, but $\begin{bmatrix}4\\2\end{bmatrix}$ is not an eigenvector of T. Illustrate this on the graph below by drawing two vectors: one that could be $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$, and one that could be $T\left(\begin{bmatrix}4\\2\end{bmatrix}\right)$. Be sure to label which is which.



$\begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector, so $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ must be a scalar multiple of $\begin{bmatrix}1\\2\end{bmatrix}$.

$\begin{bmatrix}4\\2\end{bmatrix}$ is not an eigenvector, so $T\left(\begin{bmatrix}4\\2\end{bmatrix}\right)$ must be a vector that is not a scalar multiple of $\begin{bmatrix}4\\2\end{bmatrix}$.