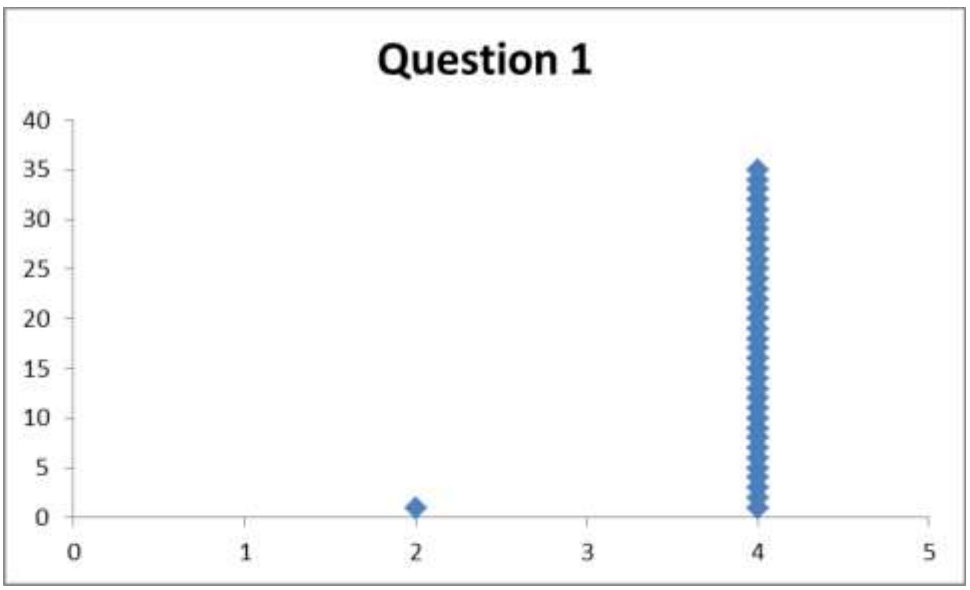
Name __Solutions____Linear Algebra, Test 3, 4/21/2014

Throughout the test simplify all answers except where stated otherwise. For questions in which the answer is a single number, word, etc, be sure to show your work or provide an explanation.

Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 1 \end{bmatrix}$

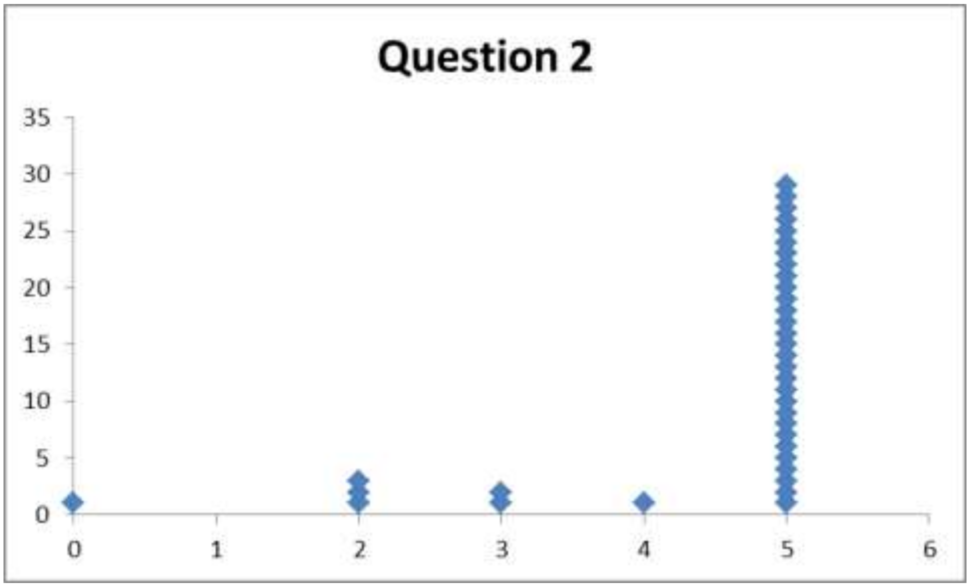
1) Find $\vec{v} \cdot \vec{w}$. (4 points)

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 4 \\ 1 \end{bmatrix} = 1 \cdot 0 + 2 \cdot 3 + (-1) \cdot 4 + 0 \cdot 1 = 6 - 4 = 2$$



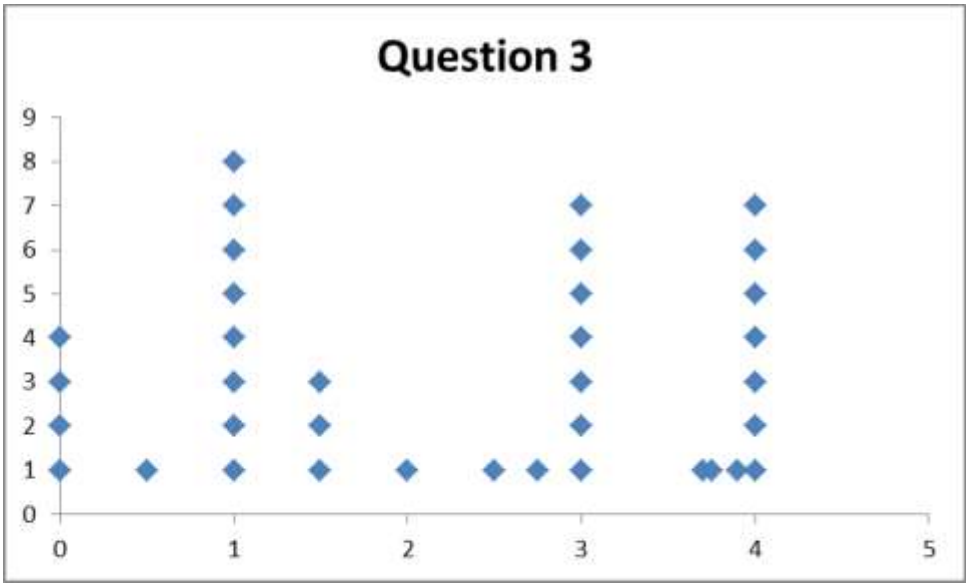
2) Find $\|\vec{v}\|$. (5 points)

$$\left\| \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2 + (-1)^2 + 0^2} = \sqrt{6}$$

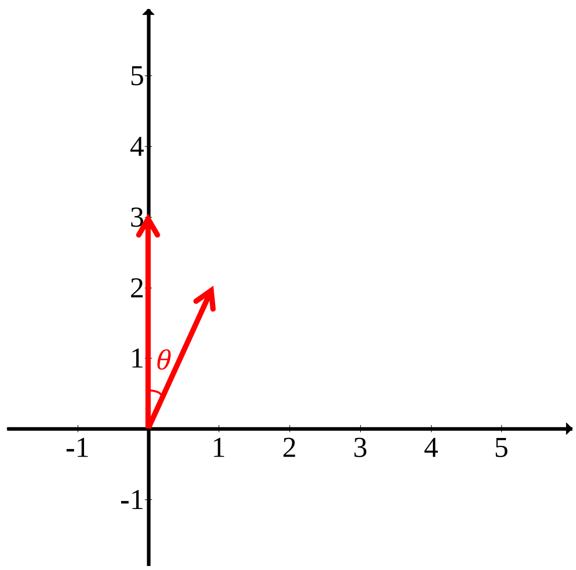


3) Find the angle between $\vec{v}$ and $\vec{w}$. No need to simplify your answer. (4 points)

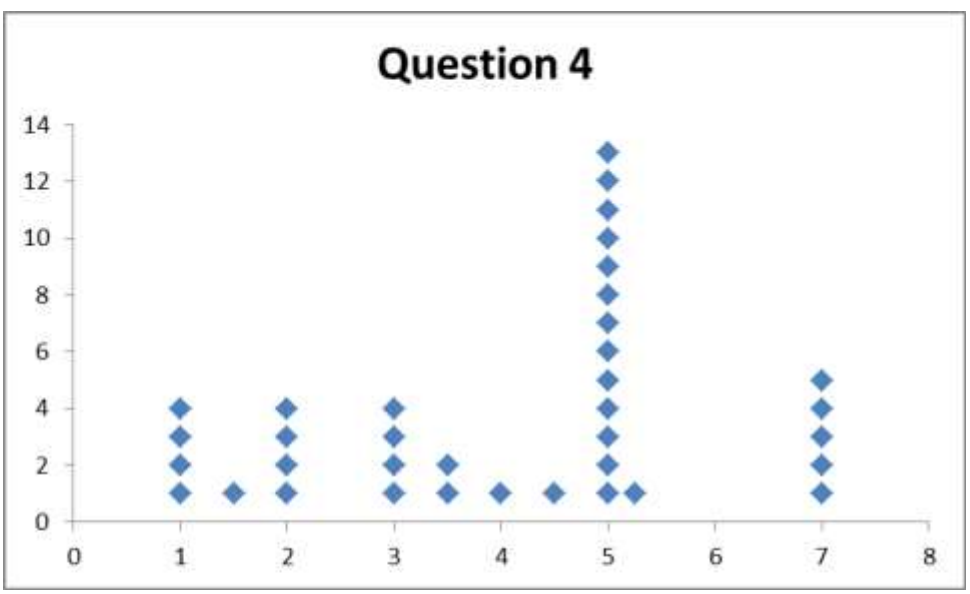
$$\cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{6}\sqrt{26}}\right)$$



4) Consider the $\mathbb{R}^2$ plane below formed by the vectors $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$. That is, $\text{span}\left(\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}\right)$ is the x axis, while $\text{span}\left(\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}\right)$ is the y-axis. Project $\vec{v}$ and $\vec{w}$ onto this plane and illustrate the angle between them on the plane. Is this angle the same as your answer to (3) above? (7 points)



The angle is not the same.



5) Diagonalize the matrix below. (20 points)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

First find the eigenvalues:

$$0 = \begin{vmatrix} 1-x & 0 & 0 & 0 \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & 1-x & 1 \\ 0 & 0 & 1 & 1-x \end{vmatrix} = (1-x)^2 \begin{vmatrix} 1-x & 1 \\ 1 & 1-x \end{vmatrix} = (1-x)^2((1-x)^2 - 1)$$

$$= (1-x)^2(x^2 - 2x) = x(x-2)(1-x)^2$$

$\lambda_1 = 0$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 2$:

$$\begin{bmatrix} 1-2 & 0 & 0 & 0 \\ 0 & 1-2 & 0 & 0 \\ 0 & 0 & 1-2 & 1 \\ 0 & 0 & 1 & 1-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda_3 = 1$:

$$\begin{bmatrix} 1-1 & 0 & 0 & 0 \\ 0 & 1-1 & 0 & 0 \\ 0 & 0 & 1-1 & 1 \\ 0 & 0 & 1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Construct the matrix P:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Find P^{-1}:

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 & 1 \end{array}\right]$$

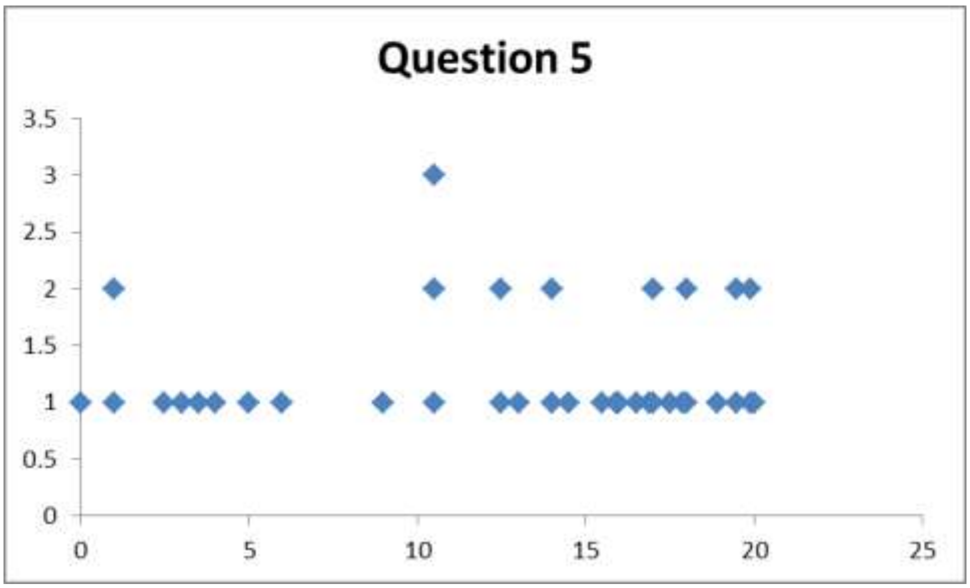
$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array}\right] \sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array}\right]$$

Hence P^{-1} is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}$$

Then put the diagonalization together:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}$$



6) Find an orthogonal basis for the space below. (10 points)

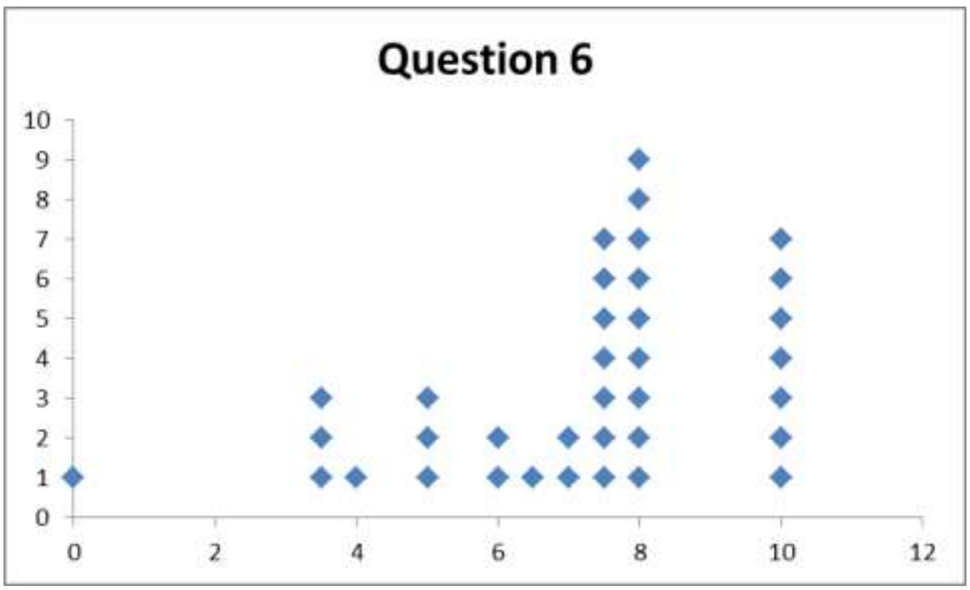
$$\text{span}\left(\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}0\\2\\2\end{bmatrix}\right\}\right)$$

First note that the third vector is dependent on the first two, so actually we can express this space as:

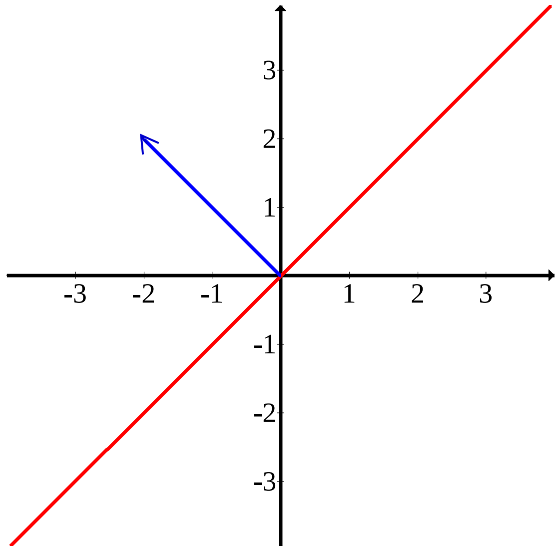
$$\text{span}\left(\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}0\\2\\2\end{bmatrix}\right\}\right) = \text{span}\left(\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}\right)$$

Now we need to find an orthogonal basis. Normally I would use the Gram-Schmidt process, but in this case the two vectors $[0 \quad 1 \quad -1]^t$ and $[0 \quad 1 \quad 1]^t$ are already orthogonal! Hence an orthogonal basis is:

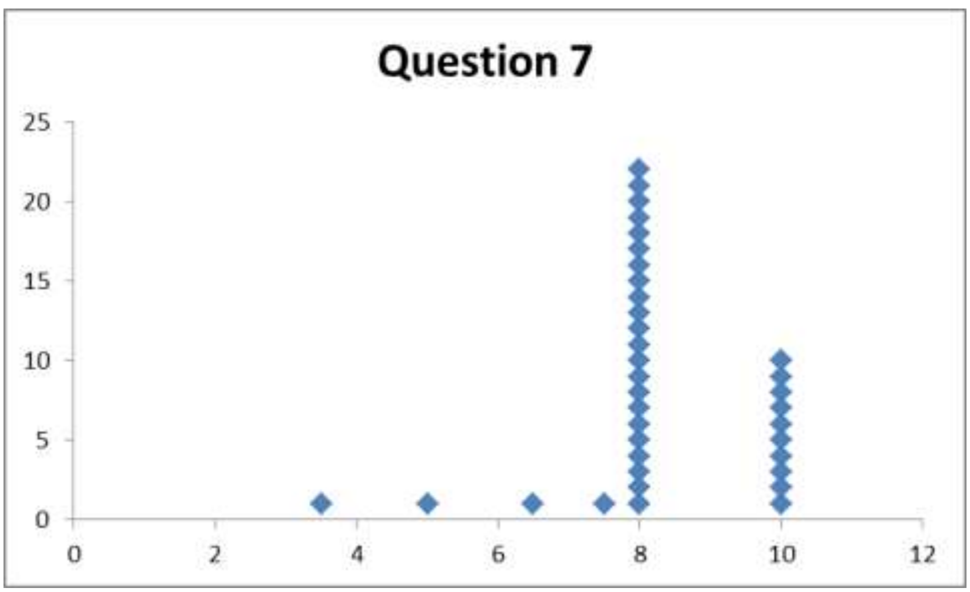
$$\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$



7) Illustrate the orthogonality between (a) the space spanned by $\begin{bmatrix}1\\1\end{bmatrix}$ and (b) a vector of your choice that is orthogonal to (a). (10 points)



Here note that we are graphing a *space* and a *vector*.

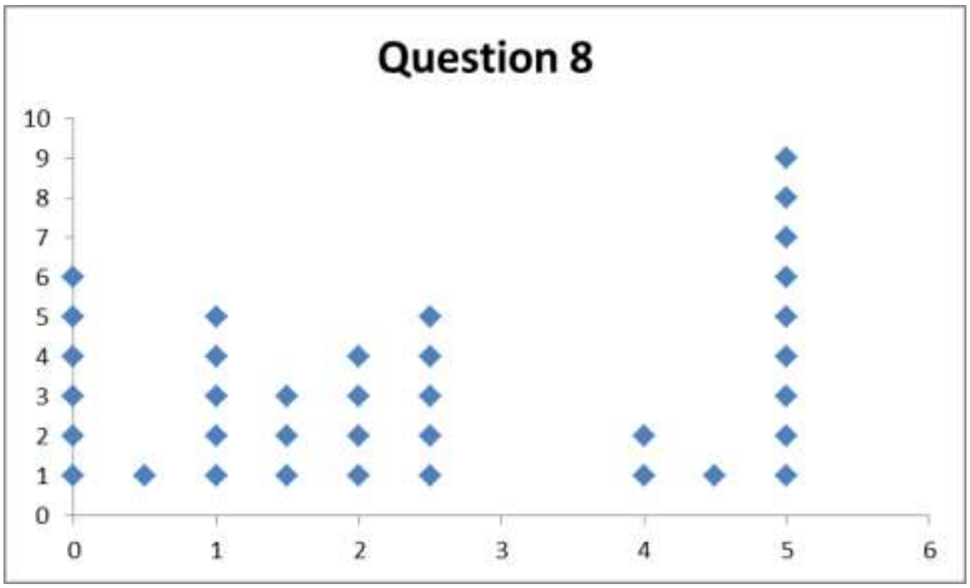


8) Find an orthogonal basis for $\mathbb{R}^{77}$. (5 points)

There are many choices, the simplest is the standard basis:

$$\left\{\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\cdots,\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}\right\}$$

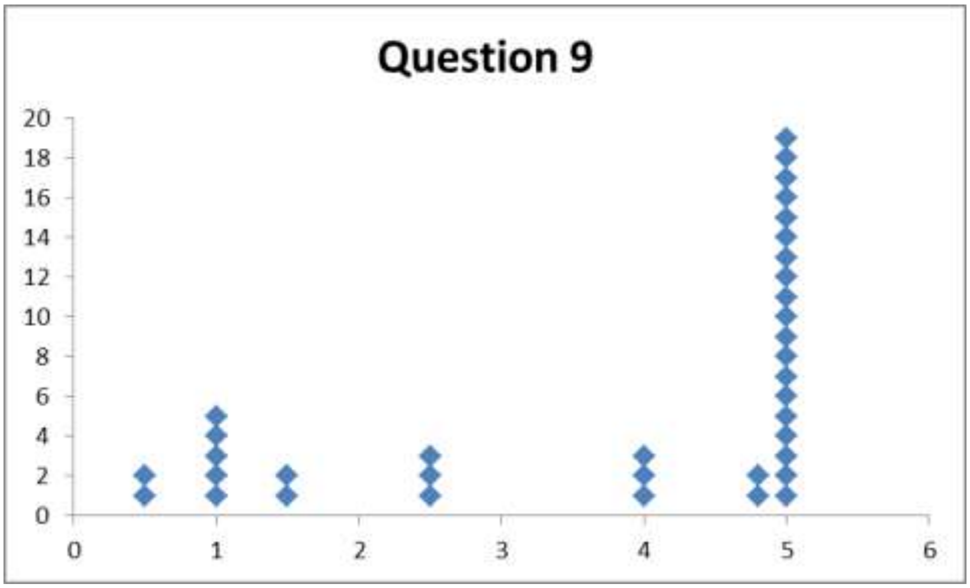
where each of the 77 vectors has 77 components with exactly one nonzero entry.



9) Determine whether or not the matrix below is diagonalizable. Justify your answer. (5 points)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

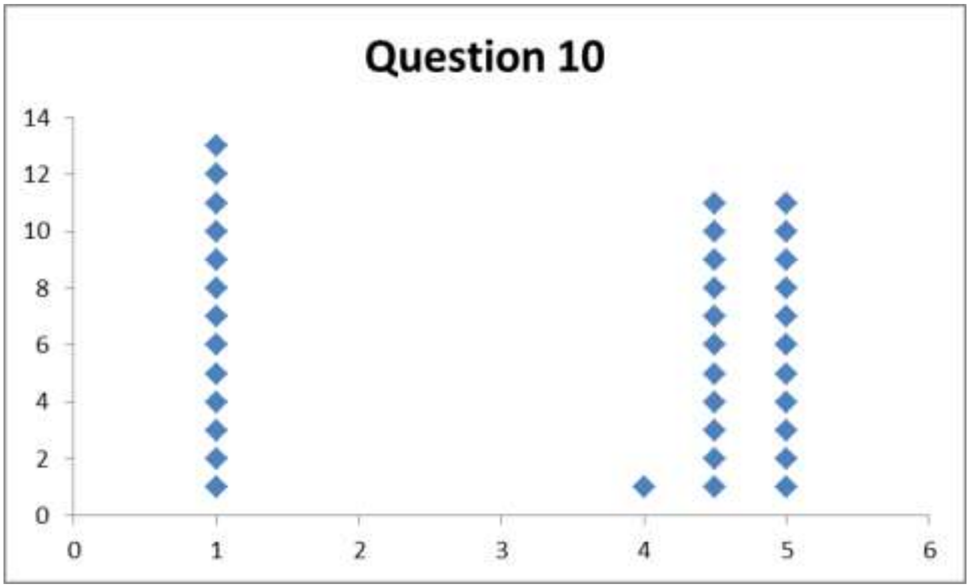
This is not diagonalizable because it does not have a complete set of eigenvectors: the only eigenvector is 1, and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a null space with dimension one.



10) Determine whether or not the matrix below exists. Justify your answer. (5 points)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{50}$$

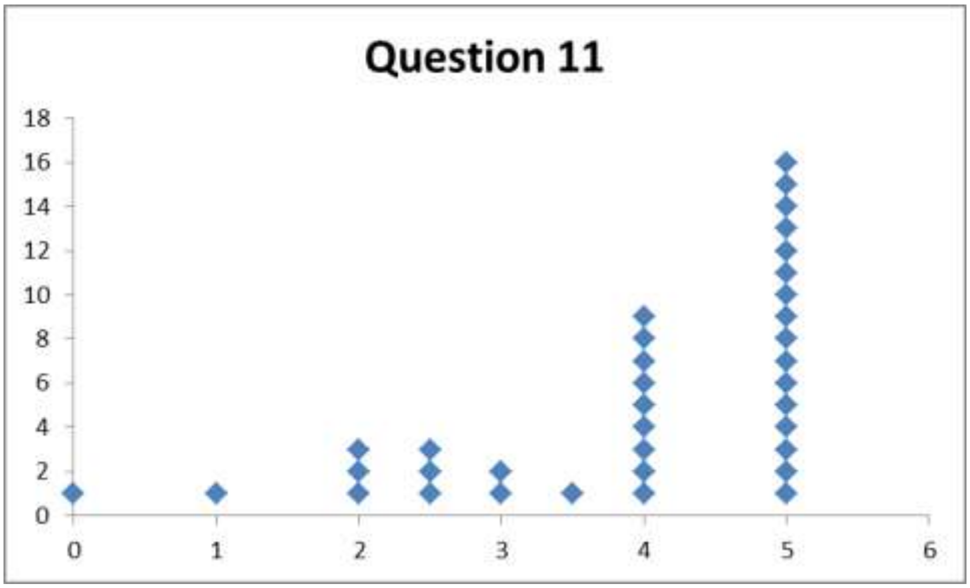
Of course it exists, it's just matrix multiplication!



11) Determine whether or not an orthogonal basis exists for the space below. Justify your answer. (5 points)

$$\text{span}\left(\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}6\\4\\7\end{bmatrix}, \begin{bmatrix}11\\2\\3\end{bmatrix}\right\}\right)$$

Yes it has an orthogonal basis because *every* space has an orthogonal basis. You can see this because you can always apply the Gram-Schmidt process.

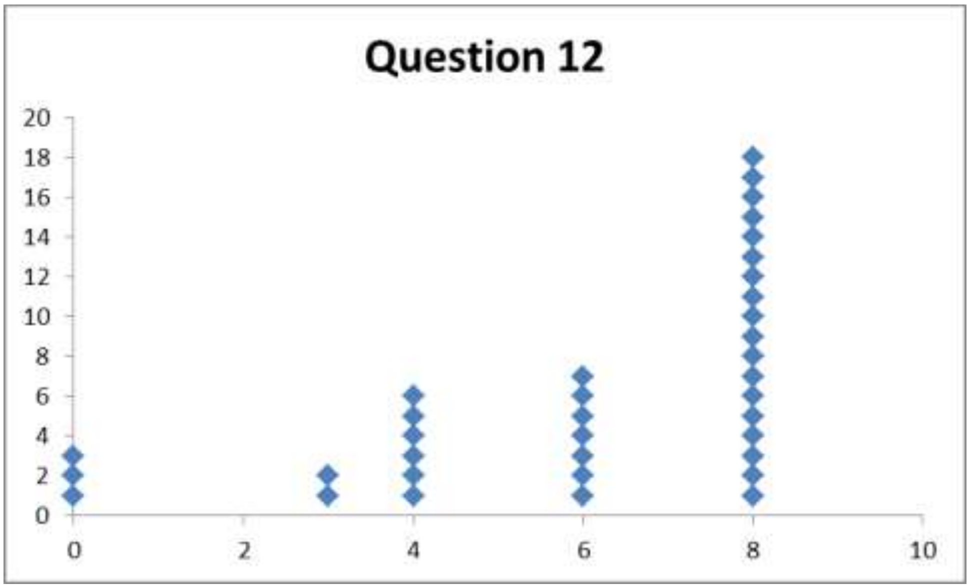


12) Find the orthogonal complement of span$\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}\right)$. (8 points)

We're looking for a one-dimensional space orthogonal to span$\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}\right)$.

The vector $\begin{bmatrix}1\\-1\end{bmatrix}$ is orthogonal to span$\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}\right)$, so let's use that as a basis and take its span:

$$\left(\text{span}\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}\right)\right)^{\perp} = \text{span}\left(\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}\right)$$



13) Express the vector $\begin{bmatrix}1\\2\\3\\-1\end{bmatrix}$ in terms of the basis $\{\vec{b}_1,\vec{b}_2,\vec{b}_3,\vec{b}_4\} = \left\{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\right\}$. (12 points)

Note that this is an orthogonal basis so we can use projections to express the vector in terms of the basis:

$$\begin{bmatrix}1\\2\\3\\-1\end{bmatrix} = \frac{-1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + \frac{2}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} + \frac{4}{2}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\\1\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$$

Or, if you like you can express it as a coordinate vector in terms of the basis $B = \{\vec{b}_1,\vec{b}_2,\vec{b}_3,\vec{b}_4\}$:

$$\begin{bmatrix}1\\2\\3\\-1\end{bmatrix} = \begin{bmatrix}-1/2\\3/2\\1\\2\end{bmatrix}_B$$