Use the system of equations below for the following FOUR problems.

\[-5x_1 - 3x_2 = 4\]
\[2x_2 = 10\]

1) Solve the system of equations.
(4 points)

The second equation gives us:
\[x_2 = 5\]

Then the first equation gives us:
\[x_1 = \frac{4 + 3x_2}{-5} = \frac{4 + 15}{-5} = -\frac{19}{5}\]
2) Write the system of equations a matrix equation $Ax = \bar{b}$.

(4 points)

$$\begin{bmatrix} -5 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$
3) Write the system of equations as a vector equation \( \vec{v}_1 c_1 + \vec{v}_2 c_2 = \vec{b} \).

(4 points)

\[
\begin{bmatrix}
-5 \\
0
\end{bmatrix} x_1 + \begin{bmatrix}
-3 \\
2
\end{bmatrix} x_2 = \begin{bmatrix}
4 \\
10
\end{bmatrix}
\]
4) Let $A$ be the $2 \times 2$ matrix with complex entries given by $A = \begin{bmatrix} 0 & a + bi \\ 0 & 0 \end{bmatrix}$. Show that $(\bar{A})^T = \bar{A}^T$.

(4 points)

\[
\begin{bmatrix} 0 & a + bi \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & a - bi \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a - bi & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 0 & a + bi \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ a + bi & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a - bi & 0 \end{bmatrix}
\]

\[
\therefore \begin{bmatrix} 0 & a + bi \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & a + bi \\ 0 & 0 \end{bmatrix}^T
\]
5) Answer the following as true or false. A statement is true if it is always true; false if it is ever false. (Assume sizes are such that addition and multiplication operations make sense) (2 points each)

T  F  a) Suppose $A$ is a square matrix with two equal columns. Then $A$ is invertible.
T  F  b) Suppose $A$ and $B$ are square matrices. Then $AB = BA$.
T  F  c) Suppose $A$ and $B$ are square matrices. Then $A^2 - B^2 = (A - B)(A + B)$.
T  F  d) Suppose $A$ is not a square matrix. Then $A$ is symmetric.
T  F  e) Suppose $A$ is not a square matrix. Then $A^T A$ is symmetric.
T  F  f) Suppose $A$ and $C$ are square matrices, but $B$ is not. Then $(ABC)^T = C^T B^T A^T$.
T  F  g) Suppose $A$, $B$, and $C$ are invertible matrices. Then $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$.
6) Find $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 4 \end{bmatrix}^7$

(4 points)

\[
\begin{bmatrix} 2 & 0 \\ -1 & 5 \\ 3 & 4 \end{bmatrix}
\]

Question 6 $r=0.473$
7) Find an example of a $2 \times 4$ matrix $A$ such that $AA^T$ is nonsingular.

(6 points)

This question is a bit tricky/tedious. If it hadn’t been a homework problem I’d of left more time for you to try to figure it out on the test. It turns out there’s a really simple solution, though, after you’ve figured out what to do:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(There are multiple possible other answers)
8) Find a formula for the quadratic form \( q = q(x, y, z) \) with the matrix representation
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & -2
\end{bmatrix}.
\]

(4 points)

\[x^2 + 3y^2 - 2z^2\]

Full credit was also given if you used \( x_1, x_2, \) and \( x_3, \) or correctly expressed it as a matrix expression.
9) Given the matrix $A$ below, does the nonhomogeneous system of equations $A\vec{x} = \vec{b}$ have a solution for every choice of $b$? Justify your answer. (6 points)

$A = \begin{bmatrix}
1 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 0 & 2 & 4
\end{bmatrix}$

Yes it does, because in reduced row echelon form we see that there is a pivot in each row which allows us to solve any system of equations using this matrix:

$\begin{bmatrix}
1 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2
\end{bmatrix}$

Question 9 $r=0.519$
10) Partition the matrix below so that it has 6 parts.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 2 & 2 & 2 \\
3 & 6 & 9 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(4 points)

(There are multiple possible other answers)
11) Let $A, B, C$, and $X$ be matrices of appropriate sizes. Assume everything is invertible. Solve the equation $B(X + A)^{-1} = C$ for $X$. Show your work. (6 points)

\[
(X + A)^{-1} = B^{-1}C \\
X + A = C^{-1}B \\
X = C^{-1}B - A
\]
12) Determine if \[
\begin{bmatrix}
3 \\
1 \\
2
\end{bmatrix}
\] and \[
\begin{bmatrix}
2 \\
0 \\
-1
\end{bmatrix}
\] are orthogonal. Show your work.

(4 points)

\[
\begin{bmatrix}
3 \\
1 \\
2
\end{bmatrix} \cdot \begin{bmatrix}
2 \\
0 \\
-1
\end{bmatrix} = 6 - 2 = 4 \neq 0
\]

No, they're not orthogonal.
13) Give an example of an infinite set of vectors in $\mathbb{R}^3$ that do not form a vector space.
(4 points)

$$\left\{ \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

(There are multiple possible other answers)
14) Give an example of a vector space that is not \( \mathbb{R}^n \) for any \( n \).
(4 points)

\[ \text{span}\left( \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right) \]

(There are multiple possible other answers)
15) Find the null space of $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \end{bmatrix}$

(10 points)

The null space of $A$ is the solution set to $Ax = 0$. We can find this by row reducing the matrix:

$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \end{bmatrix} \sim R\begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & 4 \end{bmatrix}$

Now with $x_3 = a$ as a free variable, we see that $x_2 = -4a$ and $x_1 = 10a$. This gives us the solution set:

$\left\{ \begin{bmatrix} 10a \\ -4a \\ a \end{bmatrix} : a \in \mathbb{R} \right\} = \text{span}\left( \begin{bmatrix} 10 \\ -4 \\ 1 \end{bmatrix} \right)$
16) Find the inverse of $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 5 \end{bmatrix}$

(10 points)

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\sim_R \begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{5}{2} & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{5}{2} & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

**Question 16 r=0.682**
17) You know that the matrix equation $A\vec{x} = \vec{0}$ has more than one solution. What else can you say?
(4 points per insightful statement; 1 point per obvious statement; every incorrect statement nullifies a correct statement. 8 points maximum)

Some key insightful things we can say are that:

- $A$ has a column without a pivot.
- The row space of $A$ is not all of $\mathbb{R}^m$.
- $A\vec{x} = \vec{0}$ has infinitely many solutions.
- The system of equations has a free variable.