Name ____ Test 1, Fall 2017

Use the system of equations below for the following FOUR problems.

$$-5x_1 - 3x_2 = 4$$
$$2x_2 = 10$$

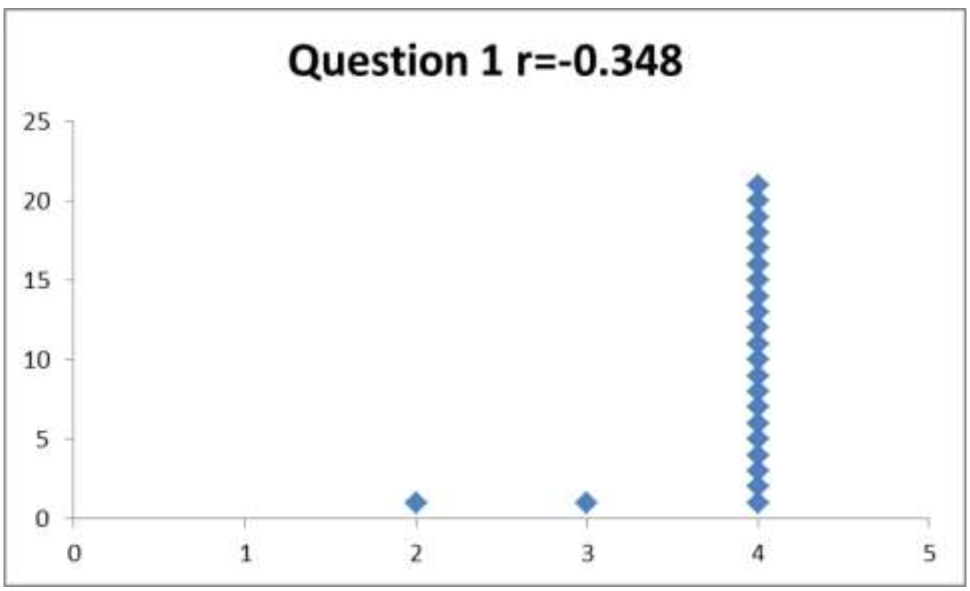
1) Solve the system of equations.
(4 points)

The second equation gives us:

$$x_2 = 5$$

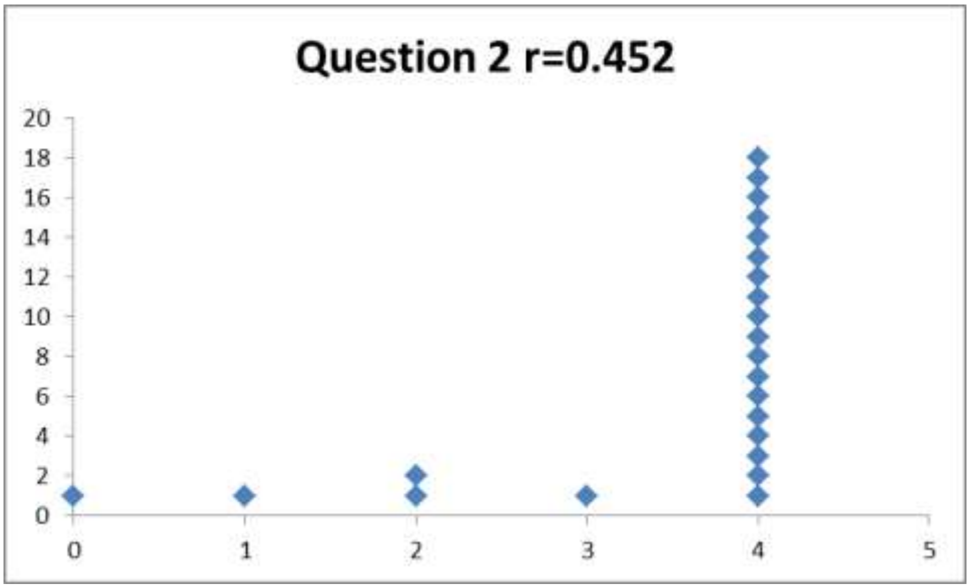
Then the first equation gives us:

$$x_1 = \frac{4 + 3x_2}{-5} = \frac{4 + 15}{-5} = -\frac{19}{5}$$



2) Write the system of equations a matrix equation $A\vec{x} = \vec{b}$.
(4 points)

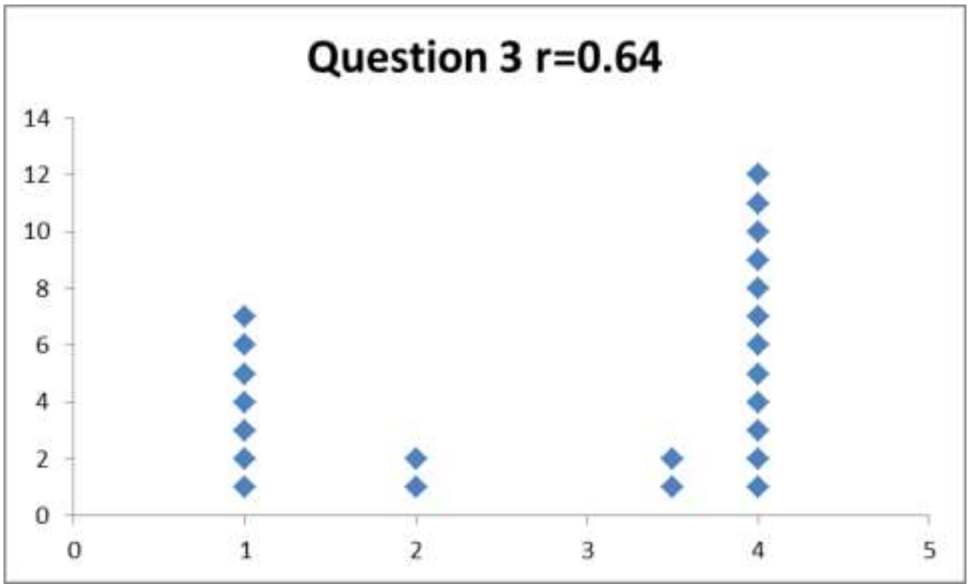
$$\begin{bmatrix} -5 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$



3) Write the system of equations as a vector equation $\vec{v}_1 c_1 + \vec{v}_2 c_2 = \vec{b}$.

(4 points)

$$\begin{bmatrix} -5 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} -3 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$



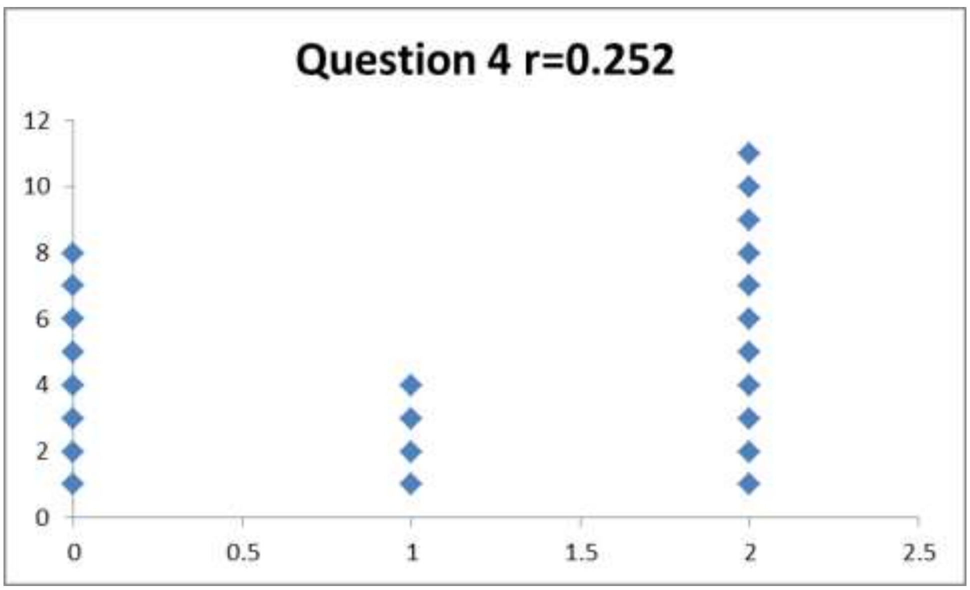
4) Let A be the 2×2 matrix with complex entries given by $A = \begin{bmatrix} 0 & a+bi \\ 0 & 0 \end{bmatrix}$. Show that $(\bar{A})^T = \overline{A^T}$.

(4 points)

$$\overline{\begin{bmatrix} 0 & a+bi \\ 0 & 0 \end{bmatrix}}^T = \begin{bmatrix} 0 & a-bi \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ a-bi & 0 \end{bmatrix}$$

$$\overline{\begin{bmatrix} 0 & a+bi \\ 0 & 0 \end{bmatrix}^T} = \overline{\begin{bmatrix} 0 & 0 \\ a+bi & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ a-bi & 0 \end{bmatrix}$$

$$\therefore \overline{\begin{bmatrix} 0 & a+bi \\ 0 & 0 \end{bmatrix}}^T = \overline{\begin{bmatrix} 0 & a+bi \\ 0 & 0 \end{bmatrix}^T}$$



5) Answer the following as true or false. A statement is true if it is *always* true; false if it is *ever* false.
(Assume sizes are such that addition and multiplication operations make sense) (2 points each)

T (F) a) Suppose A is a square matrix with two equal columns. Then A is invertible.

T (F) b) Suppose A and B are square matrices. Then $AB = BA$.

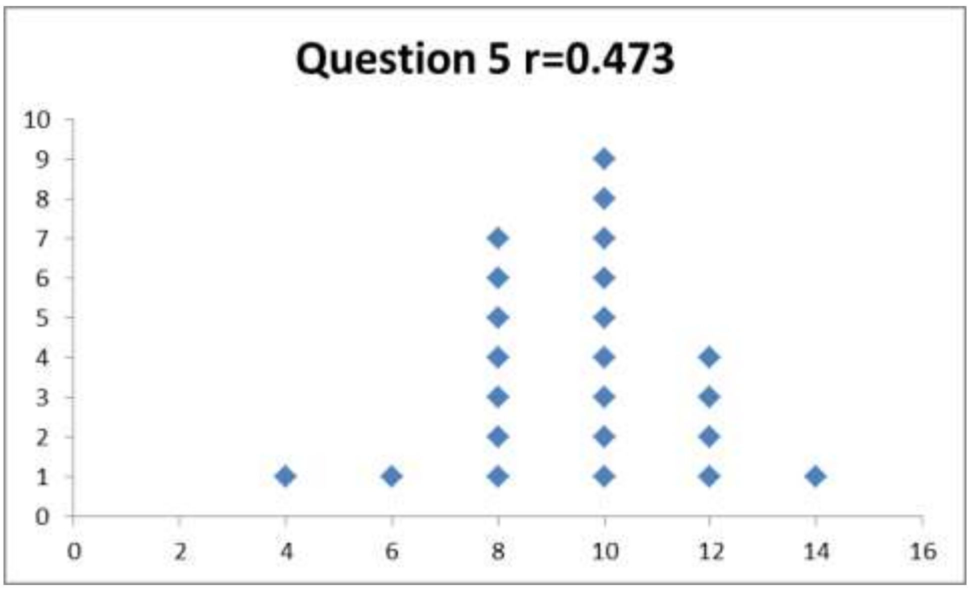
T (F) c) Suppose A and B are square matrices. Then $A^2 - B^2 = (A - B)(A + B)$.

T (F) d) Suppose A is not a square matrix. Then A is symmetric.

(T) F e) Suppose A is not a square matrix. Then $A^T A$ is symmetric.

(T) F f) Suppose A and C are square matrices, but B is not. Then $(ABC)^T = C^T B^T A^T$.

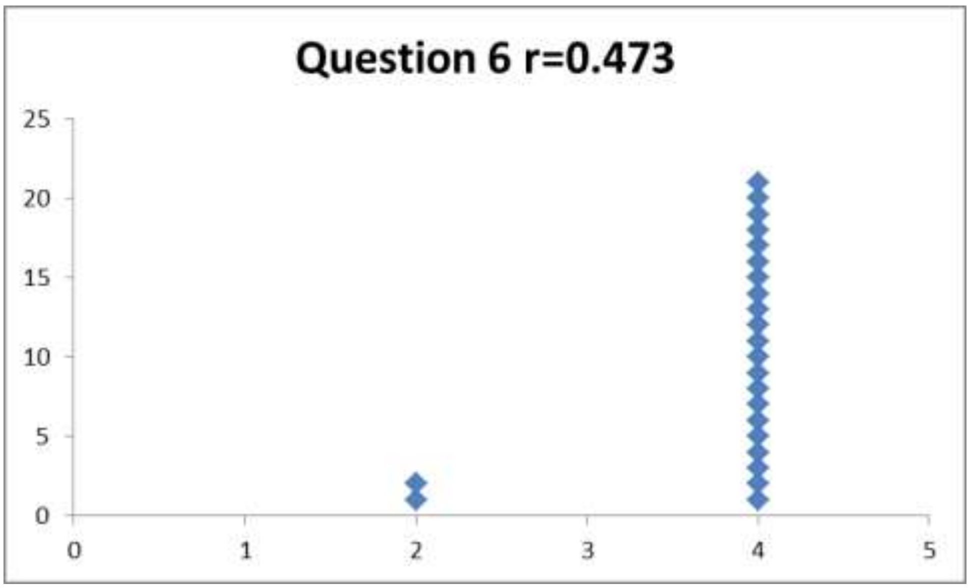
(T) F g) Suppose A, B, and C are invertible matrices. Then $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$.



6) Find $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 4 \end{bmatrix}^T$

(4 points)

$$\begin{bmatrix} 2 & 0 \\ -1 & 5 \\ 3 & 4 \end{bmatrix}$$

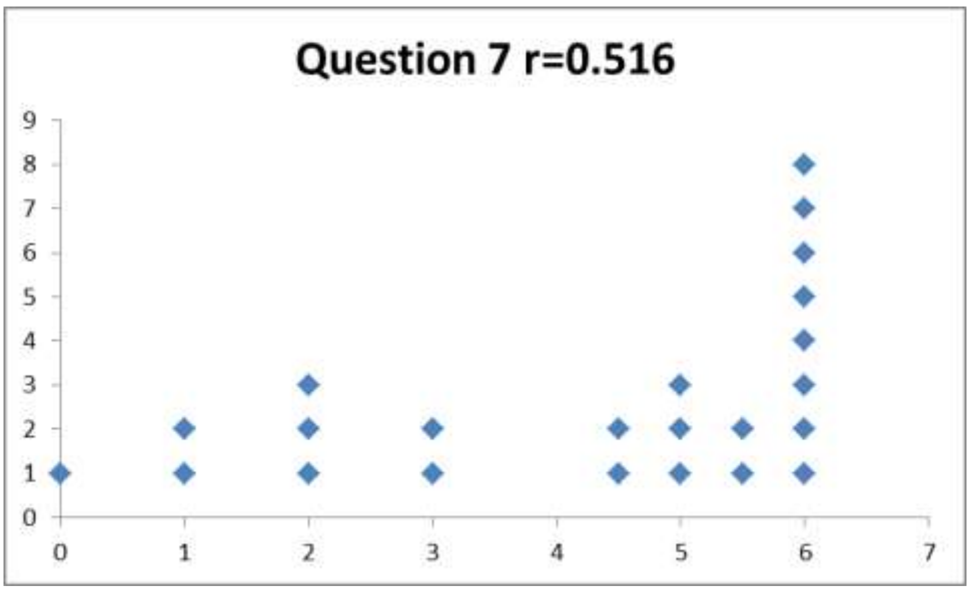


7) Find an example of a 2×4 matrix A such that AA^T is nonsingular.
(6 points)

This question is a bit tricky/tedious. If it hadn't been a homework problem I'd of left more time for you to try to figure it out on the test. It turns out there's a really simple solution, though, after you've figured out what to do:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(There are multiple possible other answers)

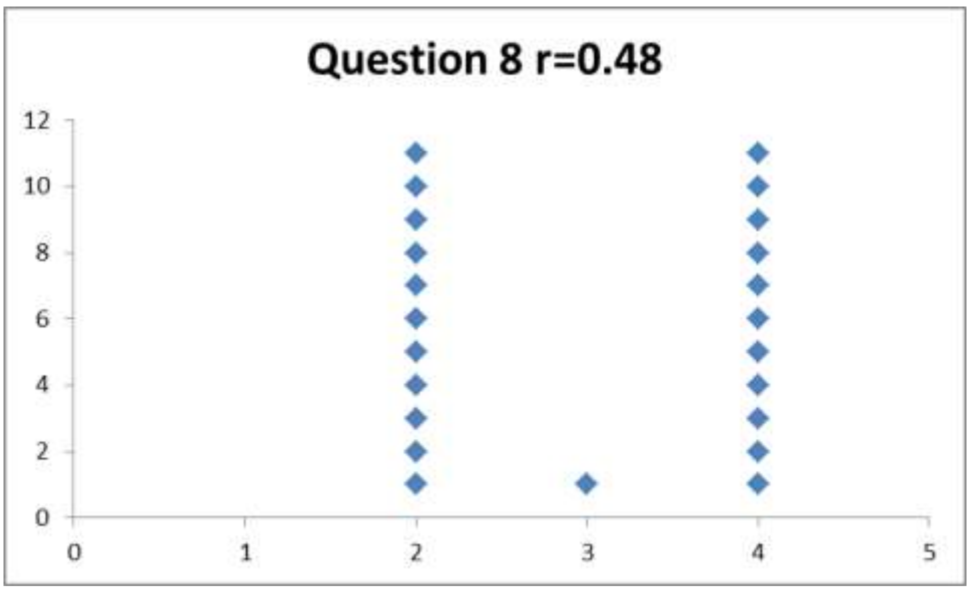


8) Find a formula for the quadratic form $q = q(x, y, z)$ with the matrix representation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

(4 points)

$$x^2 + 3y^2 - 2z^2$$

Full credit was also given if you used x_1, x_2, and x_3, or correctly expressed it as a matrix expression.

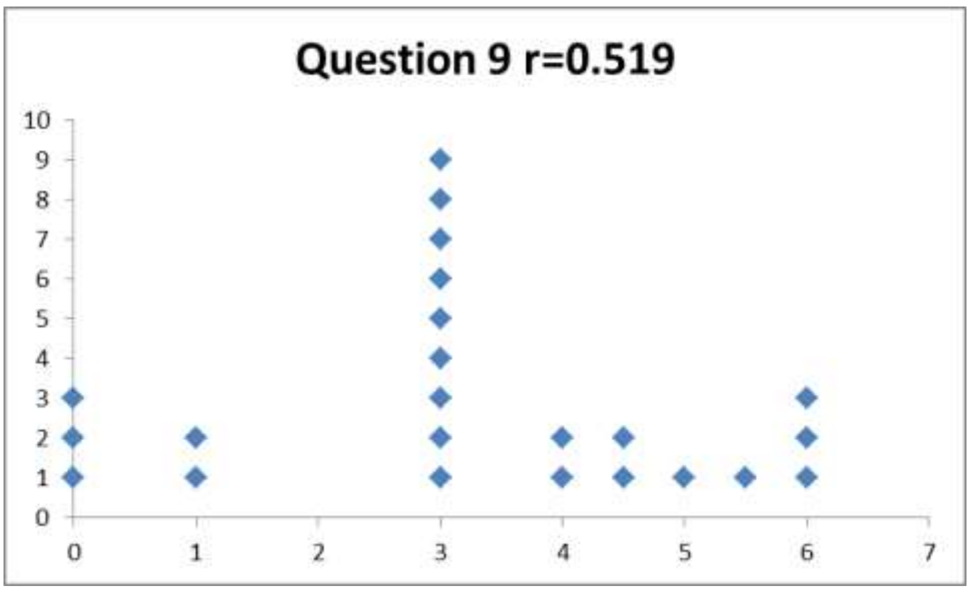


9) Given the matrix A below, does the nonhomogeneous system of equations $A\vec{x} = \vec{b}$ have a solution for every choice of b? Justify your answer. (6 points)

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

Yes it does, because in reduced row echelon form we see that there is a pivot in each row which allows us to solve any system of equations using this matrix:

$$\begin{bmatrix} \textcircled{1} & 0 & 2 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 2 \end{bmatrix}$$

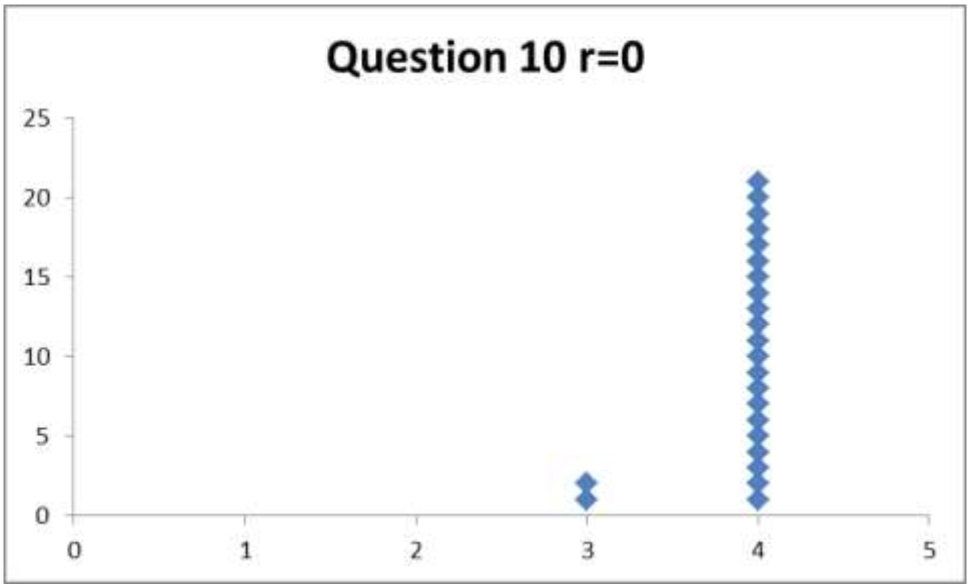


10) Partition the matrix below so that it has 6 parts.

$$\left[\begin{array}{c|cc|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \\ 3 & 6 & 9 & 0 \\ \hline 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

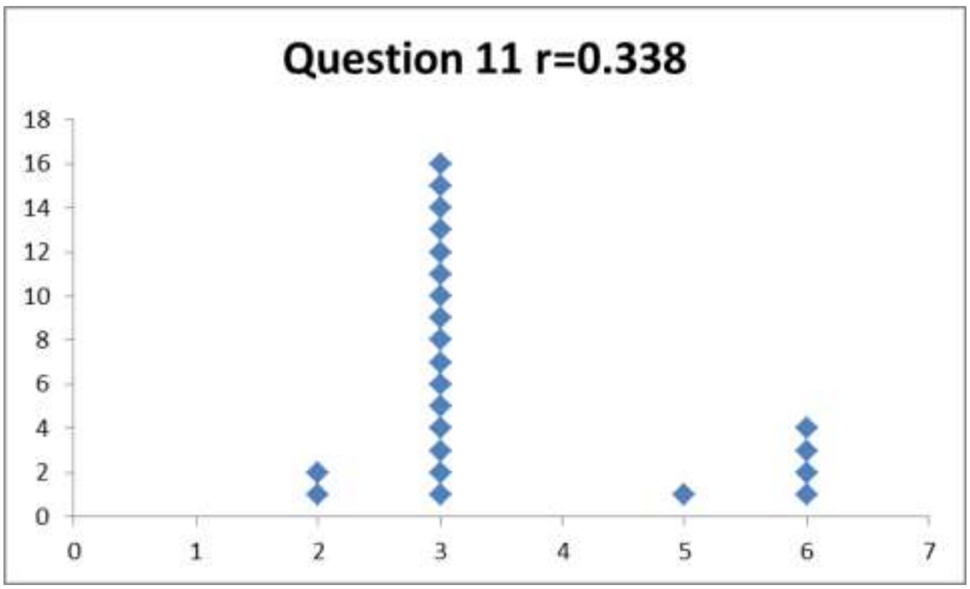
(4 points)

(There are multiple possible other answers)



11) Let A, B, C, and X be matrices of appropriate sizes. Assume everything is invertible. Solve the equation $B(X + A)^{-1} = C$ for X. Show your work.
(6 points)

$$(X + A)^{-1} = B^{-1}C$$
$$X + A = C^{-1}B$$
$$X = C^{-1}B - A$$

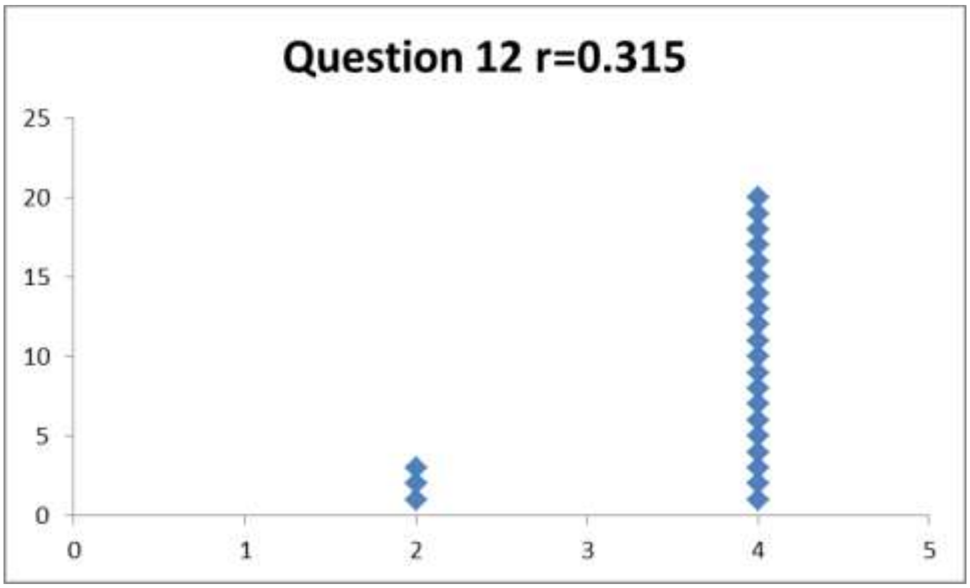


12) Determine if $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ are orthogonal. Show your work.

(4 points)

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = 6 - 2 = 4 \neq 0$$

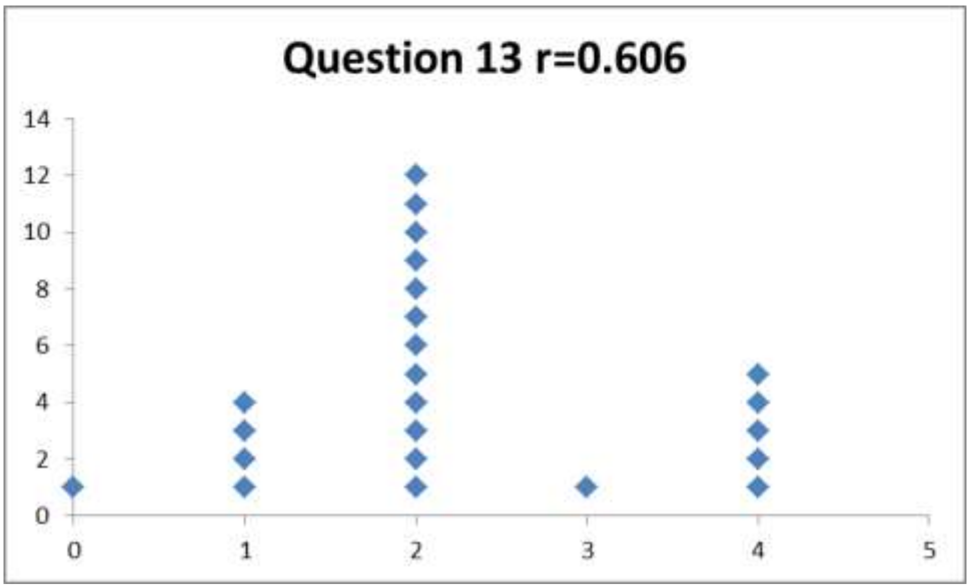
No, they're not orthogonal.



13) Give an example of an infinite set of vectors in $\mathbb{R}^3$ that do not form a vector space.
(4 points)

$$\left\{ \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

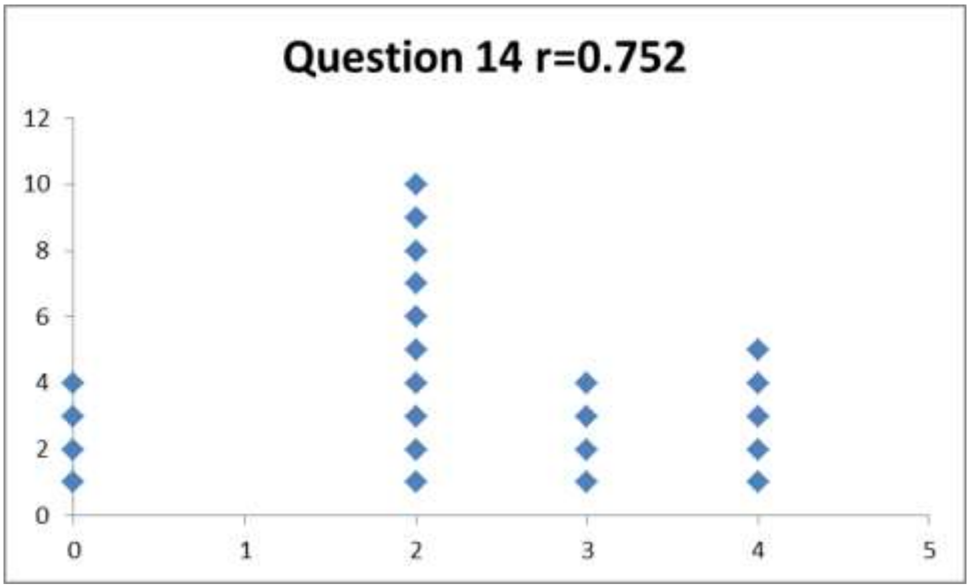
(There are multiple possible other answers)



14) Give an example of a vector space that is not $\mathbb{R}^n$ for any n.
(4 points)

$$\text{span}\left(\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}\right)$$

(There are multiple possible other answers)



15) Find the null space of $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \end{bmatrix}$

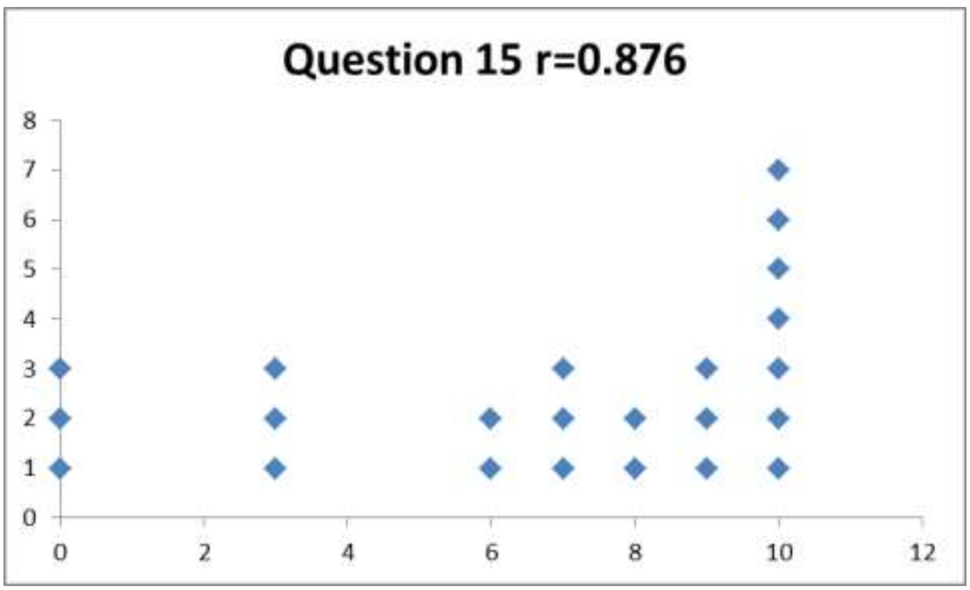
(10 points)

The null space of A is the solution set to $A\vec{x} = \vec{0}$. We can find this by row reducing the matrix:

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & 4 \end{bmatrix}$$

Now with $x_3 = a$ as a free variable, we see that $x_2 = -4a$ and $x_1 = 10a$. This gives us the solution set:

$$\left\{ \begin{bmatrix} 10a \\ -4a \\ a \end{bmatrix} : a \in \mathbb{R} \right\} = \text{span}\left(\left\{ \begin{bmatrix} 10 \\ -4 \\ 1 \end{bmatrix} \right\} \right)$$



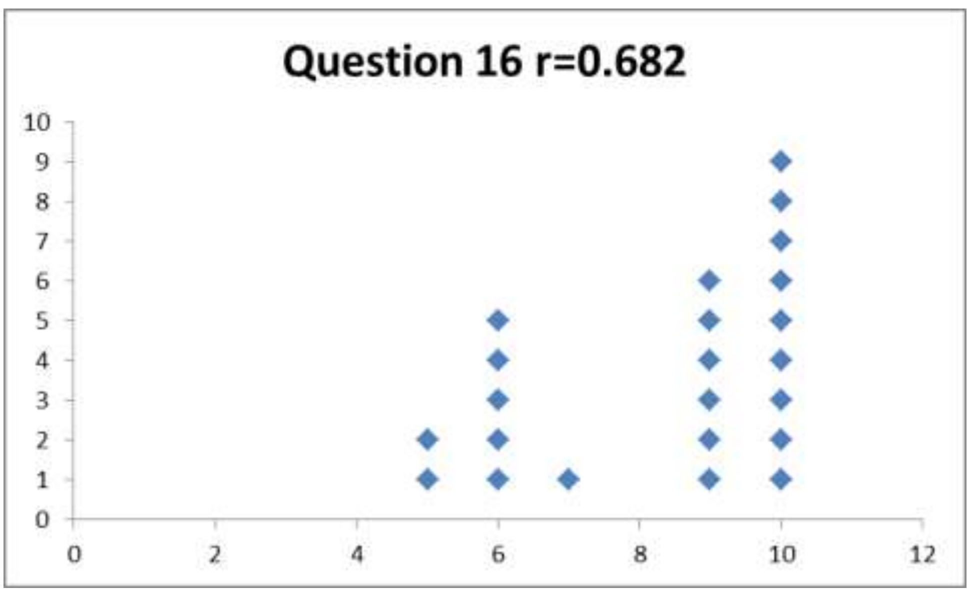
16) Find the inverse of $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 5 \end{bmatrix}$

(10 points)

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

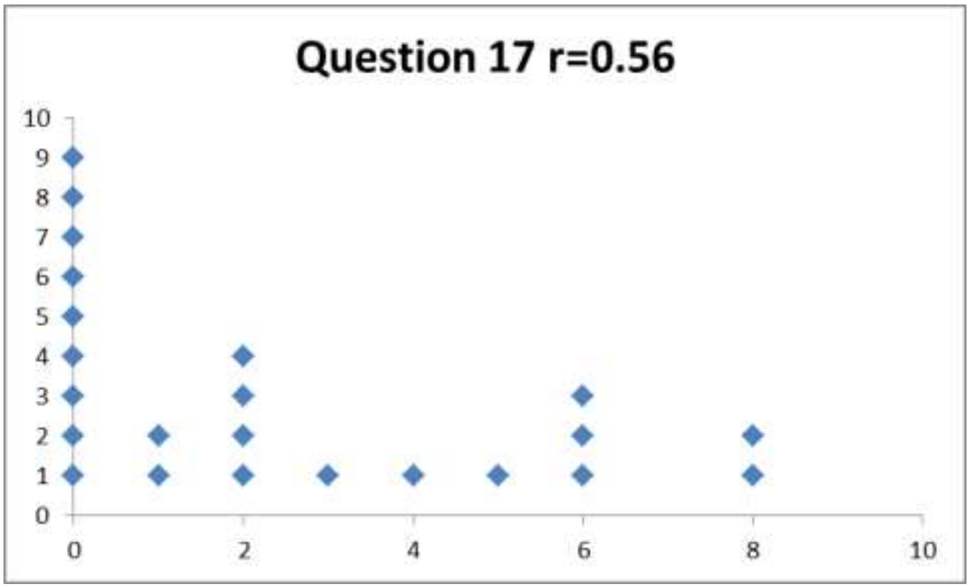
$$\sim_R \begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{5}{2} & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 5 \end{bmatrix}^{=1} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{5}{2} & -2 \\ 0 & -1 & 1 \end{bmatrix}$$



17) You know that the matrix equation $A\vec{x} = \vec{0}$ has more than one solution. What else can you say?
(4 points per insightful statement; 1 point per obvious statement; every incorrect statement nullifies a correct statement. 8 points maximum)

Some key insightful things we can say are that:
A has a column without a pivot.
The row space of A is not all of $\mathbb{R}^m$.
$A\vec{x} = \vec{0}$ has infinitely many solutions.
The system of equations has a free variable.



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