Non-calculator portion. Turn this portion in before you take out your technology.

1) An equation \( A\vec{x} = \vec{b} \) has the solution given below, where \( s_1 \) and \( s_2 \) are free variables. Express the solution as a meaningful linear combination of vectors.

\[
\begin{align*}
x_1 &= -1 + 2s_1 \\
x_2 &= 3s_1 + 2s_2 \\
x_3 &= s_1 - s_2
\end{align*}
\]

(6 points)

2) Expand the set below to give a basis for \( \mathbb{R}^2 \).

\[
\begin{bmatrix} 1 \\ -3 \end{bmatrix}
\]

(4 points)

3) A \( 8 \times 5 \) matrix has a null space of dimension 3. What is the rank of \( A \)?

(6 points)
4) Suppose the matrix $A$ has size $11 \times 7$. That is, 11 rows and 7 columns. It is known that the equation $A\vec{x} = \vec{0}$ has multiple solutions, but that $A\vec{x} = [1 \ 2 \ 3 \ 3 \ 1]^T$ has no solutions. Find the smallest possible dimension the null space of $A$ could have.

(6 points)
5) Find the determinant of the matrix below.

\[
\begin{bmatrix}
-4 & 2 & -2 \\
1 & 0 & 1 \\
3 & -1 & 1 \\
\end{bmatrix}
\]

(10 points)
6) Find the matrix $A$ so that the linear transformation below is given by $T(\vec{x}) = A\vec{x}$.

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 
\end{bmatrix} \mapsto \begin{bmatrix}
  x_2 - 5x_3 \\
  7x_3 
\end{bmatrix}
\]

(6 points)

7) Let $B$ be defined below. Find the change of basis matrix $[I]_B^S$.

$$B = \begin{bmatrix}
  1 & 2 & 0 \\
  2 & 4 & 2 \\
  3 & 6 & 5 
\end{bmatrix}$$

(4 points)
8) Answer the following as true or false. A statement is true if it is *always* true; false if it is *ever* false.

(2 points each)

T  F  a) Every matrix $A$ has a determinant.

T  F  b) The range, $T(V)$, of a linear transformation $T: V \rightarrow W$ is a vector space.

T  F  c) If a linear transformation $T: V \rightarrow W$ is one-to-one, then it is onto.

T  F  d) If $A\vec{x} = \vec{b}$ can be solved for every choice of $\vec{b}$, then the linear transformation given by $T'(\vec{x}) = A\vec{x}$ is onto.

T  F  e) Using matrices, we can write down a formula for the solution $\vec{x}$ to $A\vec{x} = \vec{b}$ no matter what $\vec{b}$ is.

9) Let $T'$ be the linear transformation below. Prove that $T'$ is onto by either using some theorem/concept or by finding the input $\vec{x}$ that maps to the output $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

$T': \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$

(4 points)
10) Let $A$ be an $n \times n$ matrix. You know that the matrix equation $Ax = 0$ has more than one solution. What else can you say?

(4 points per insightful statement; 1 point per obvious statement; every incorrect statement nullifies a correct statement. 12 points maximum)
Technology portion: After you turn in the non-calculator portion, you may take out your technology and finish this portion. If you’re using a phone or tablet, leave it flat on the desk.

Use the matrix $A$, below, for the problems on this page.

\[
A = \begin{bmatrix}
1 & 1 & 2 & -1 & 1 \\
1 & 2 & 1 & 0 & 2 \\
-1 & -4 & 1 & -2 & 3 \\
1 & -2 & 5 & -4 & -2
\end{bmatrix}
\]

11) Find the null space of $A$.
(6 points)

12) Is \[
\begin{bmatrix}
1 \\
2 \\
3 \\
-2
\end{bmatrix}
\]
in the vector space spanned by \[
\begin{bmatrix}
1 \\
1 \\
-1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
-4 \\
1
\end{bmatrix}, \begin{bmatrix}
2 \\
1 \\
5 \\
-4
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
-2 \\
-4
\end{bmatrix}
\]? Why or why not?
(6 points)
13) Find a subset of the vectors below that is a basis of $\mathbb{R}^3$

\[
\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \right\}
\]

(6 points)

14) Let $B_1$ and $B_2$ be defined below. Find the change of basis matrix $[I]_{B_1}^{B_2}$.

$B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right\}$, $B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(8 points)
15) Define the linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ via $T(\vec{x}) = A\vec{x}$. What is $T([1 \quad 2 \quad 4 \quad 2]^T)$?

$A = \begin{bmatrix}
6 & 1 & 2 & 1 \\
3 & 1 & 0 & 0 \\
0 & 2 & 2 & 1
\end{bmatrix}$

(6 points)

16) Define the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^5$ via $T(\vec{x}) = A\vec{x}$ where $A$ is defined below. Is $T$ one-to-one? Why or why not?

$A = \begin{bmatrix}
1 & 2 & 3 \\
3 & 2 & 1 \\
0 & 1 & 2 \\
0 & 3 & 6 \\
1 & 1 & 1
\end{bmatrix}$

(6 points)