Name ____ Test 2, Fall 2017

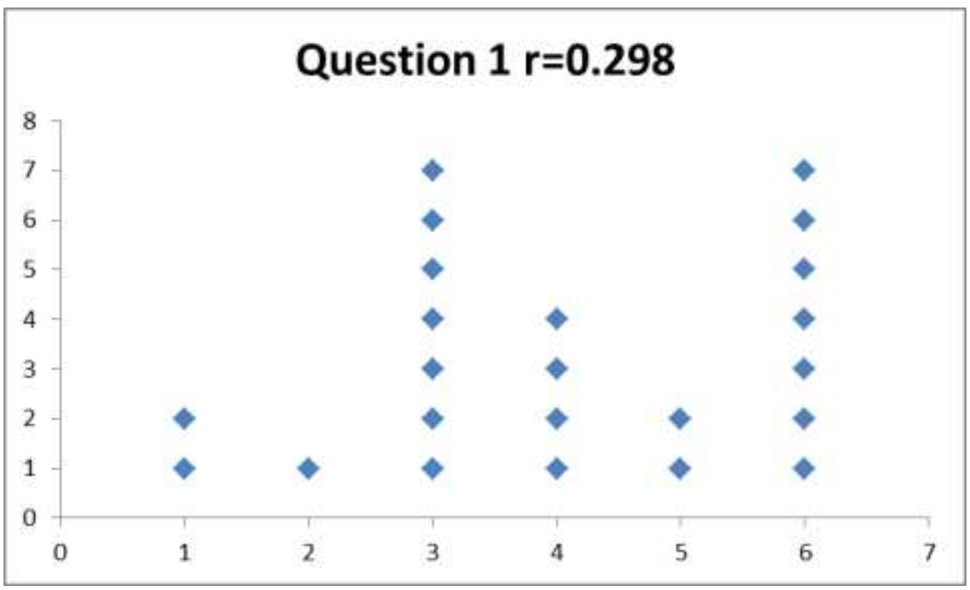
Non-calculator portion. Turn this portion in before you take out your technology.

1) An equation $A\vec{x} = \vec{b}$ has the solution given below, where s_1 and s_2 are free variables. Express the solution as a meaningful linear combination of vectors.

$$x_1 = -1 + 2s_1$$
$$x_2 = 3s_1 + 2s_2$$
$$x_3 = s_1 - s_2$$

(6 points)

$$\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} s_1 + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} s_2$$

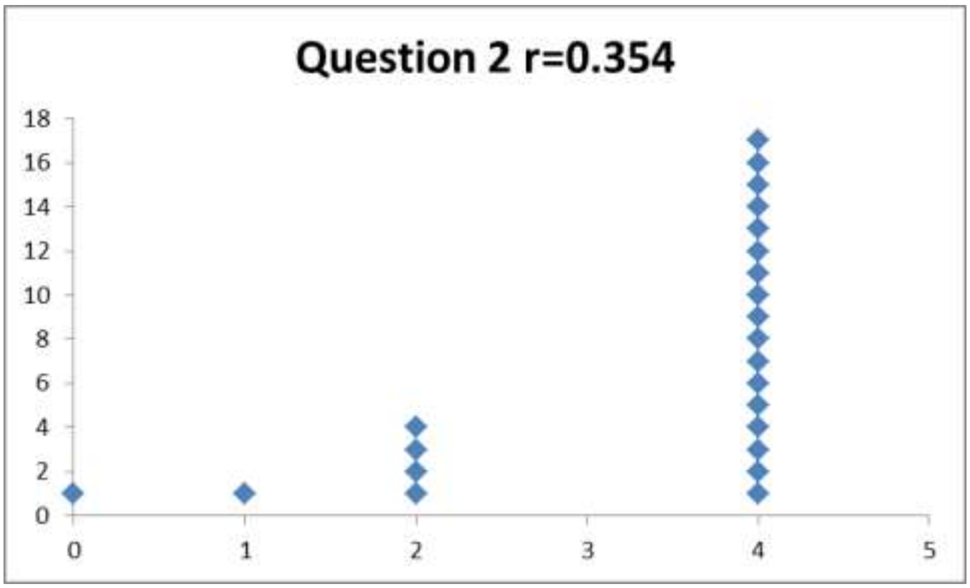


2) Expand the set below to give a basis for $\mathbb{R}^2$.

$$\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$$

(4 points)

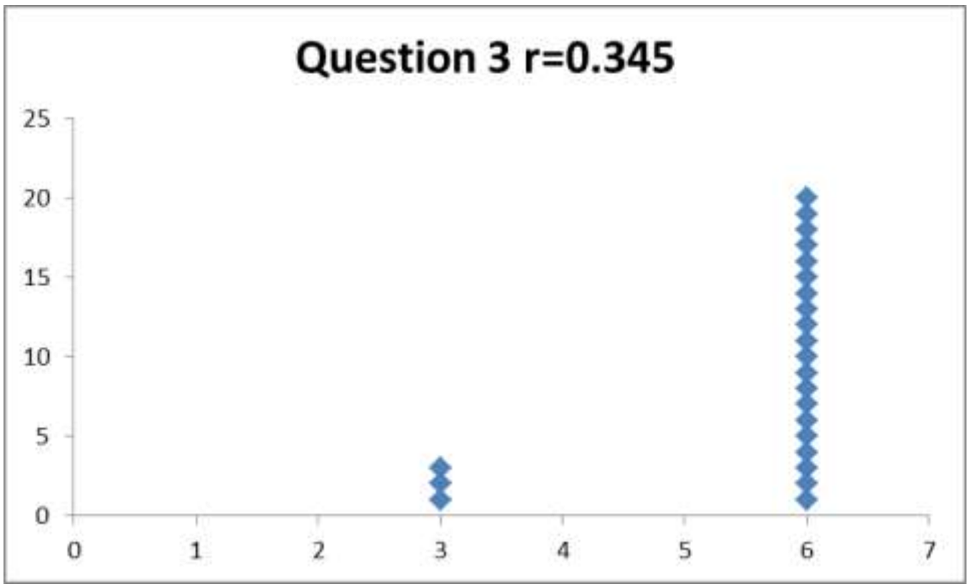
$$\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$



3) A 8×5 matrix has a null space of dimension 3. What is the rank of A?
(6 points)

2

The fact that the nullity is 3 tell us that there are 3 free variables. There are 5 columns to play with, and so the other two have pivots – that gives us our rank.



4) Suppose the matrix A has size 11×7. That is, 11 rows and 7 columns. It is known that the equation $A\vec{x} = \vec{0}$ has multiple solutions, but that $A\vec{x} = [1 \quad 2 \quad 3 \quad 3 \quad 1]^T$ has no solutions. Find the *smallest* possible dimension the null space of A could have.
(6 points)

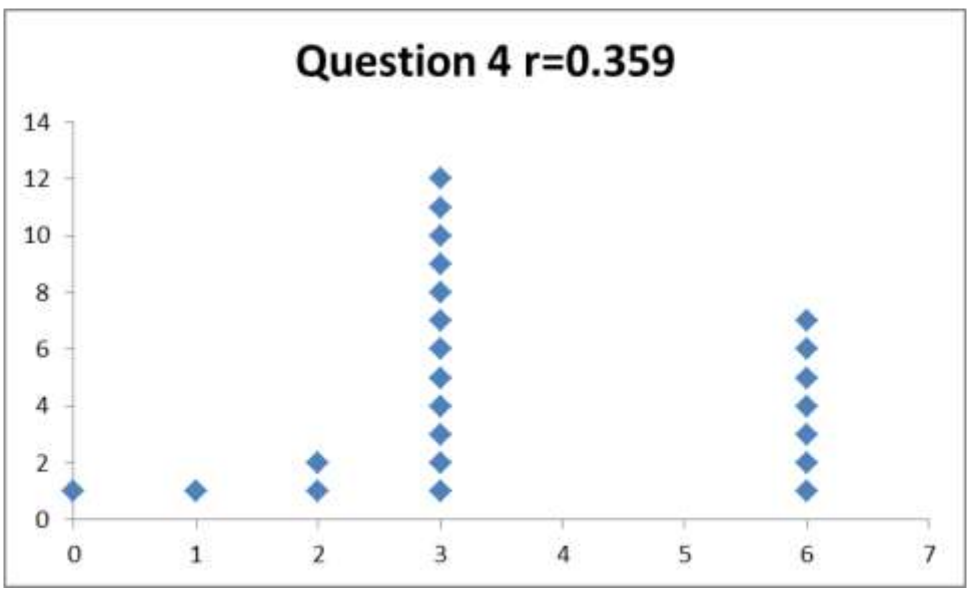
Note: The $\vec{b}$ should be replaced with a vector in $\mathbb{R}^{11}$ because A has 11 rows. Otherwise the problem doesn't really make sense because $A\vec{x}$ and $\vec{b}$ aren't the same size.

7 columns means the null space could have dimension 0, 1, 2, 3, 4, 5, 6, or 7.

$A\vec{x} = \vec{0}$ has multiple solutions means there must be at least one free variable, so the null space can't have dimension 0.

$A\vec{x} = [1 \quad 2 \quad 3 \quad 3 \quad 1]^T$ has no solutions tell us that there must be a zero row in reduced row echelon form. But there are more rows than columns, so this is a bit of a red herring and tells us nothing about the null space.

The smallest dimension of the null space is 1.



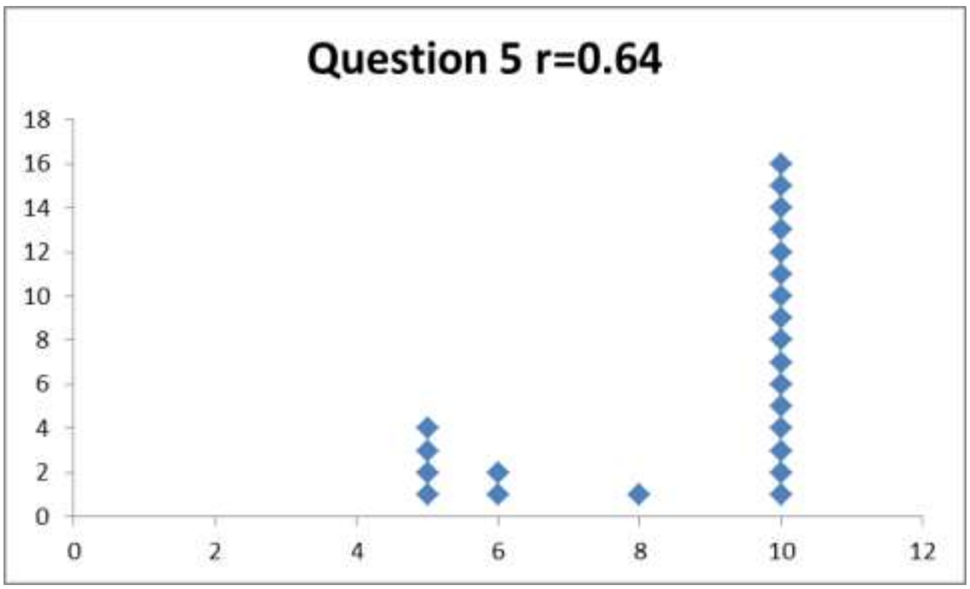
5) Find the determinant of the matrix below.

$$\begin{bmatrix} -4 & 2 & -2 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

(10 points)

Expanding it on the second row we get:

$$-\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -4 & 2 \\ 3 & -1 \end{vmatrix} = -(2-2) - (4-6) = 2$$

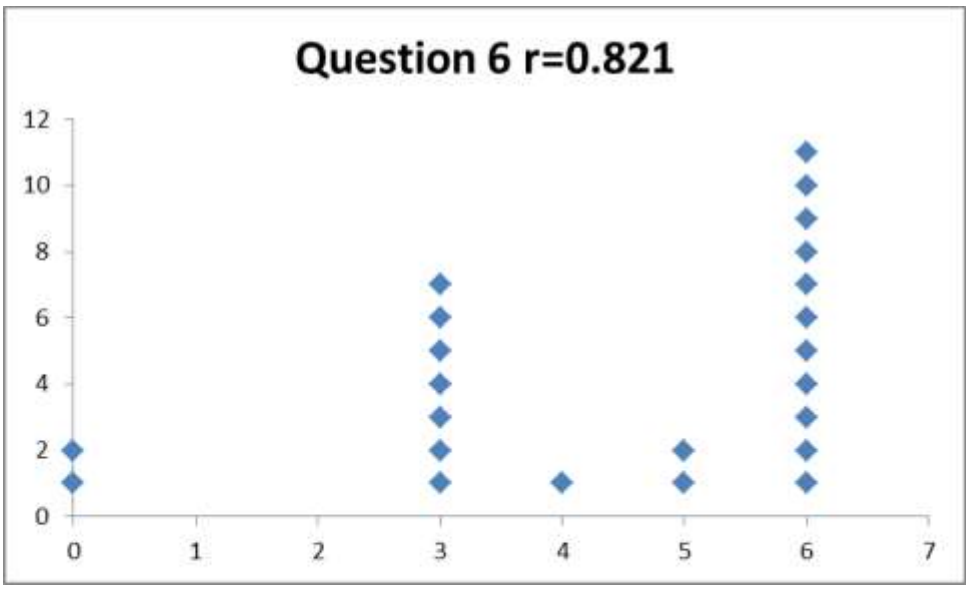


6) Find the matrix A so that the linear transformation below is given by $T(\vec{x}) = A\vec{x}$.

$$T: \mathbb{R}^3 \to \mathbb{R}^2$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_2 - 5x_3 \\ 7x_3 \end{bmatrix}$$

(6 points)

$$A = \begin{bmatrix} 0 & 1 & -5 \\ 0 & 0 & 7 \end{bmatrix}$$

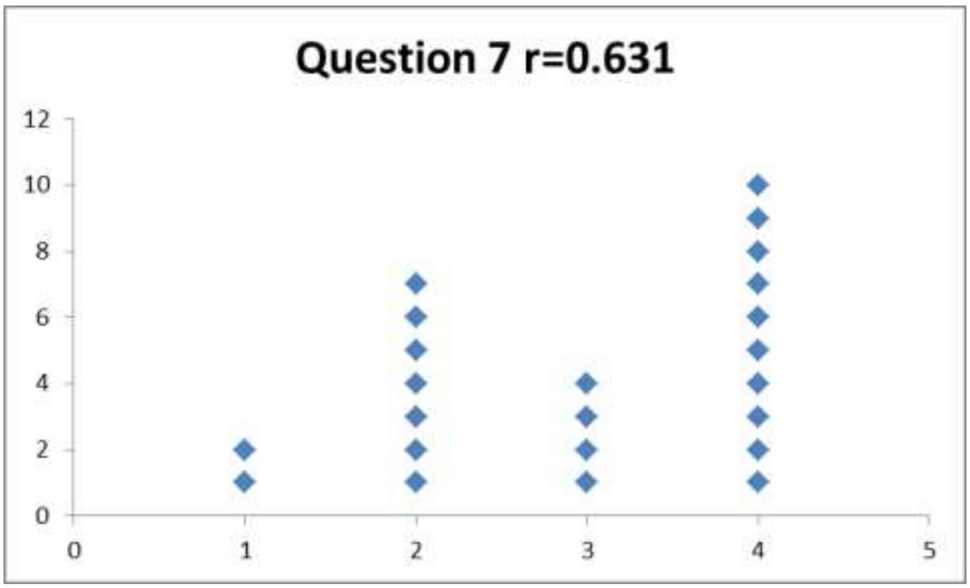


7) Let B be defined below. Find the change of basis matrix $[I]_B^S$.

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right\}$$

(4 points)

$$[I]_B^S = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 4 & 2 \\ 3 & 6 & 5 \end{bmatrix}$$



8) Answer the following as true or false. A statement is true if it is *always* true; false if it is *ever* false.
(2 points each)

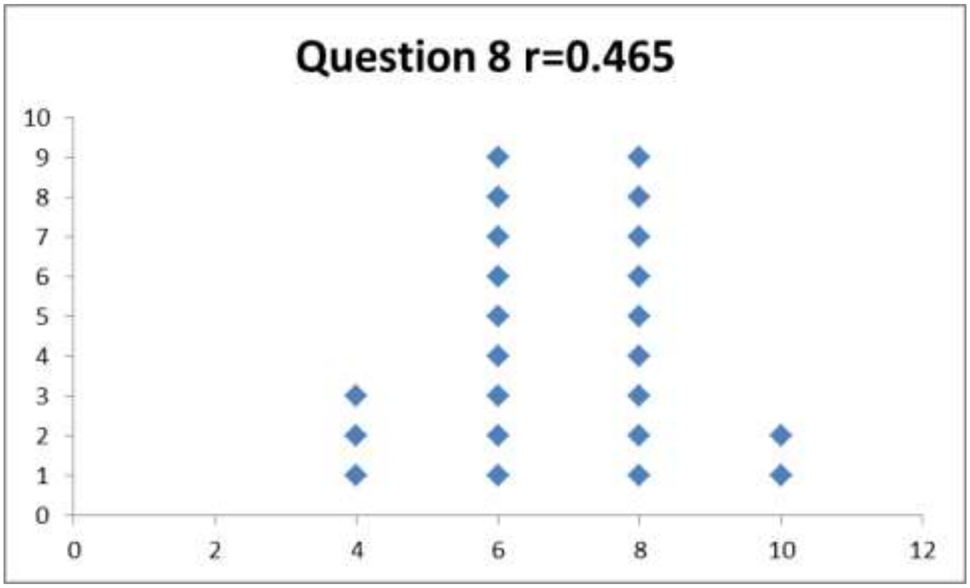
T (F) a) Every matrix A has a determinant.

(T) F b) The range, $T(V)$, of a linear transformation $T: V \to W$ is a vector space.

T (F) c) If a linear transformation $T: V \to W$ is one-to-one, then it is onto.

(T) F d) If $A\vec{x} = \vec{b}$ can be solved for every choice of $\vec{b}$, then the linear transformation given by $T(\vec{x}) = A\vec{x}$ is onto.

T (F) e) Using matrices, we can write down a formula for the solution $\vec{x}$ to $A\vec{x} = \vec{b}$ no matter what $\vec{b}$ is.



9) Let T be the linear transformation below. Prove that T is onto by either using some theorem/concept or by finding the input $\vec{x}$ that maps to the output $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

$$T: \mathbb{R}^3 \to \mathbb{R}^2$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$$

(4 points)

Using a theorem/concept:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has a pivot in each row. Hence T is onto.

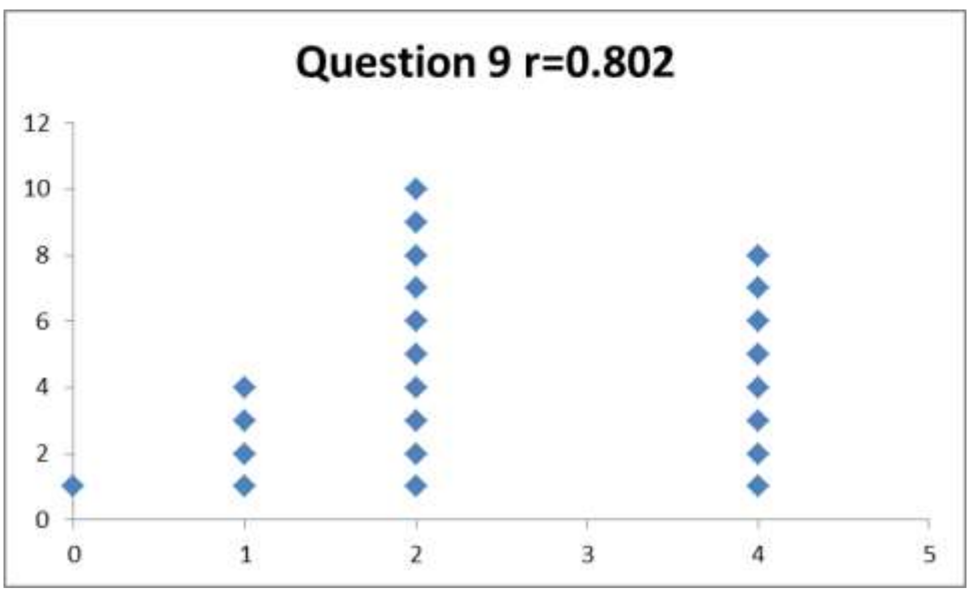
Doing it directly:

Suppose $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$

(Do scratch work elsewhere to figure out what $\vec{x}$ should be on the next line)

Choose $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ \frac{1}{2} y_2 \\ 0 \end{bmatrix}$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} y_1 \\ \frac{1}{2} y_2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ 2\left(\frac{1}{2} y_2\right) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$



10) Let A be an $n \times n$ matrix. You know that the matrix equation $A\vec{x} = \vec{0}$ has more than one solution. What else can you say?

(4 points per insightful statement; 1 point per obvious statement; every incorrect statement nullifies a correct statement. 12 points maximum)

The fact that there is more than one solution means that we have the negation of everything in the big theorem:

- A is not invertible.
- A is singular.
- $A\vec{x} = \vec{0}$ has more than the trivial solution. (Can't use this one – it was what was given!!!)
- A is not a product of elementary matrices.
- Not every row has a pivot. (In reduced row echelon form)
- Not every column has a pivot. (In reduced row echelon form)
- $rank(A) < n$
- The rows of A are linearly dependent
- $RS(A) \subset \mathbb{R}^n$ (proper subset)
- $\dim(RS(A)) < n$
- The columns of A are linearly dependent
- $CS(A) \subset \mathbb{R}^n$ (proper subset)
- $\dim(CS(A)) < n$
- $|A| = 0$

Color scheme:

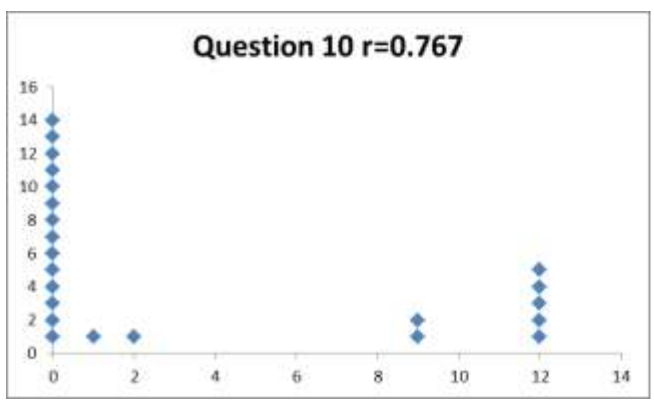
Green ink – your statement doesn't make sense. I don't mean it's wrong or right, it's impossible to evaluate whether it's right or wrong because the grammar literally doesn't make sense. Think about it as a compile error in a computer program: we can't evaluate if the logic is right or wrong because there isn't any output.

If the grammar is close enough to correct that I can make sense of it, half credit was awarded. Denoted with a circled "2".

Blue ink – incorrect statements that cancelled a correct statement.

Pink ink – Something can be correctly said, but more information is needed. Likely an ambiguous pronoun, such as "it has infinitely many solutions". What is "it"?

Orange ink – Things that are just too obvious to give any points for. Such as "A nas n columns." Yeah. I told you that!



Technology portion: After you turn in the non-calculator portion, you may take out your technology and finish this portion. If you're using a phone or tablet, leave it flat on the desk.

Use the matrix A, below, for the problems on this page.

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 & 1 \\ 1 & 2 & 1 & 0 & 2 \\ -1 & -4 & 1 & -2 & 3 \\ 1 & -2 & 5 & -4 & -2 \end{bmatrix}$$

11) Find the null space of A.
(6 points)

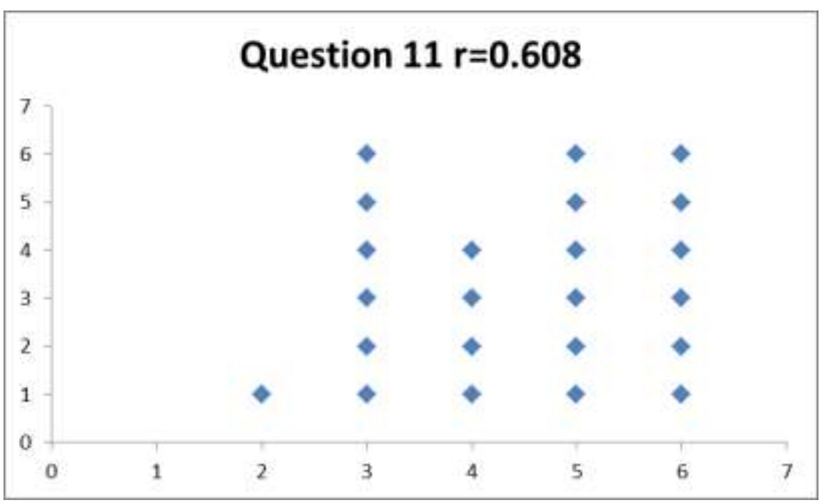
$$\begin{bmatrix} 1 & 1 & 2 & -1 & 1 \\ 1 & 2 & 1 & 0 & 2 \\ -1 & -4 & 1 & -2 & 3 \\ 1 & -2 & 5 & -4 & -2 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_5 = 0$$

x_4 is free – let's choose $x_4 = b$
x_3 is free – let's choose $x_3 = a$

$$x_2 = a - b$$
$$x_1 = -3a + 2b$$

$$NS(A) = \left\{ \begin{bmatrix} -3a + 2b \\ a - b \\ a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$



12) Is $\begin{bmatrix}1\\2\\3\\-2\end{bmatrix}$ in the vector space with basis $\left\{\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\2\\-4\\-2\end{bmatrix},\begin{bmatrix}2\\1\\1\\5\end{bmatrix},\begin{bmatrix}-1\\0\\-2\\-4\end{bmatrix}\right\}$? Why or why not?

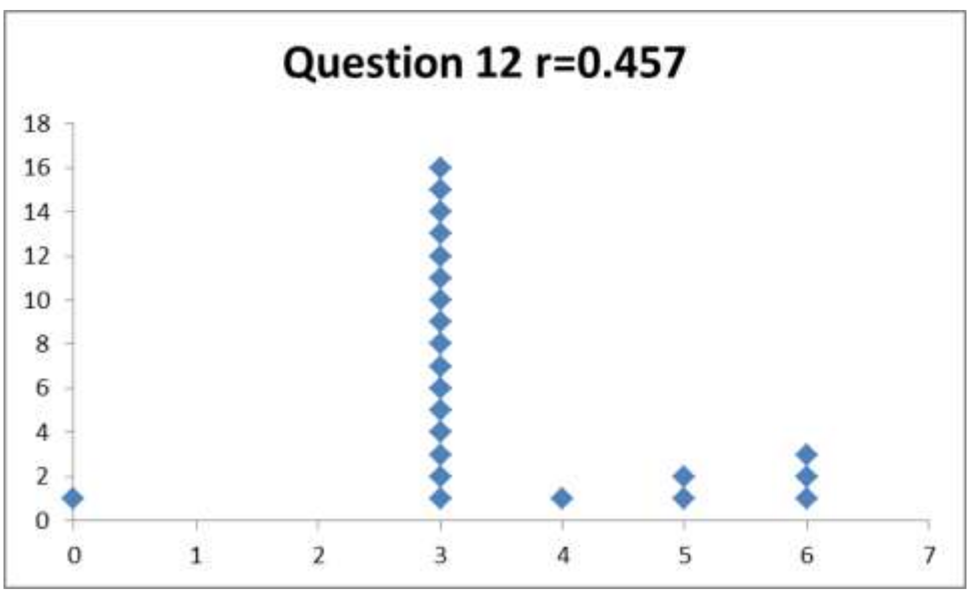
(6 points)

I should have said the vector space "spanned by" these and not used the word basis. They're not linearly independent, so they can't be a basis. Full credit was given retroactively if you said yes, because 4 vectors that form a basis in $\mathbb{R}^4$ generate everything in $\mathbb{R}^4$ including that given vector.

Here be careful, the matrix to row reduce is actually the 4×5 matrix after we've added in the extra vector. If you only reduce the original, you lose information compared with the extra vector.

$$\begin{bmatrix}1 & 1 & 2 & -1 & 1\\1 & 2 & 1 & 0 & 2\\-1 & -4 & 1 & -2 & 3\\1 & -2 & 5 & -4 & -2\end{bmatrix} \sim_R \begin{bmatrix}1 & 0 & 3 & -2 & 0\\0 & 1 & -1 & 1 & 0\\0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

From the pivots, we see that the fifth column is not a linear combination of the first four columns. Hence is it not in span $\left(\left\{\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\2\\-4\\-2\end{bmatrix},\begin{bmatrix}2\\1\\1\\5\end{bmatrix},\begin{bmatrix}-1\\0\\-2\\-4\end{bmatrix}\right\}\right)$.



13) Find a subset of the vectors below that is a basis of $\mathbb{R}^3$

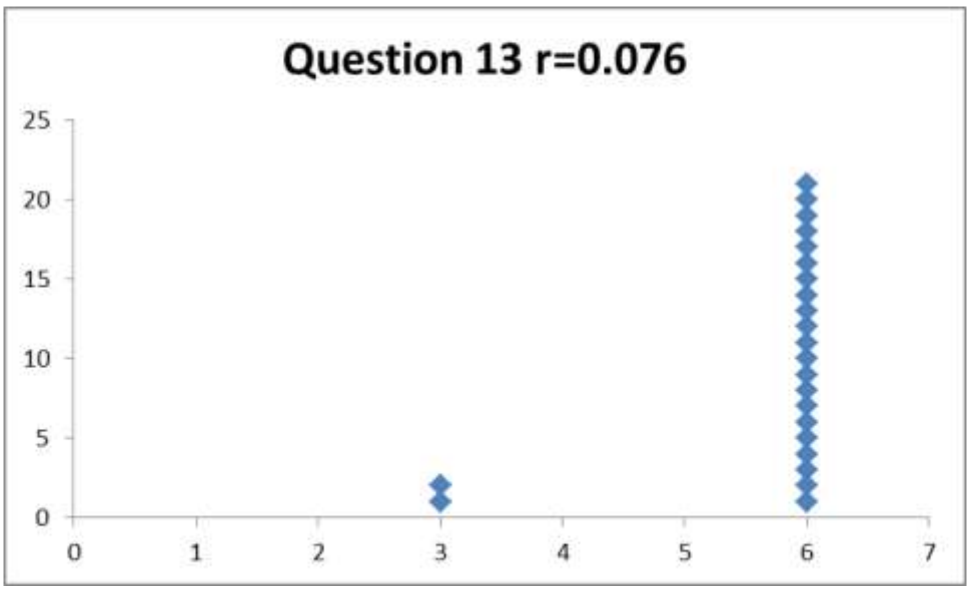
$$\left\{\begin{bmatrix}1\\2\\2\end{bmatrix},\begin{bmatrix}3\\2\\1\end{bmatrix},\begin{bmatrix}11\\10\\7\end{bmatrix},\begin{bmatrix}7\\6\\4\end{bmatrix},\begin{bmatrix}3\\2\\2\end{bmatrix}\right\}$$

(6 points)

$$\begin{bmatrix}1 & 3 & 11 & 7 & 3\\2 & 2 & 10 & 6 & 2\\2 & 1 & 7 & 4 & 2\end{bmatrix}\sim_R\begin{bmatrix}1 & 0 & 2 & 1 & 0\\0 & 1 & 3 & 2 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}$$

From the pivots we see that we need the fifth vector, and we can choose the first two:

$$\left\{\begin{bmatrix}1\\2\\2\end{bmatrix},\begin{bmatrix}3\\2\\1\end{bmatrix},\begin{bmatrix}3\\2\\2\end{bmatrix}\right\}$$

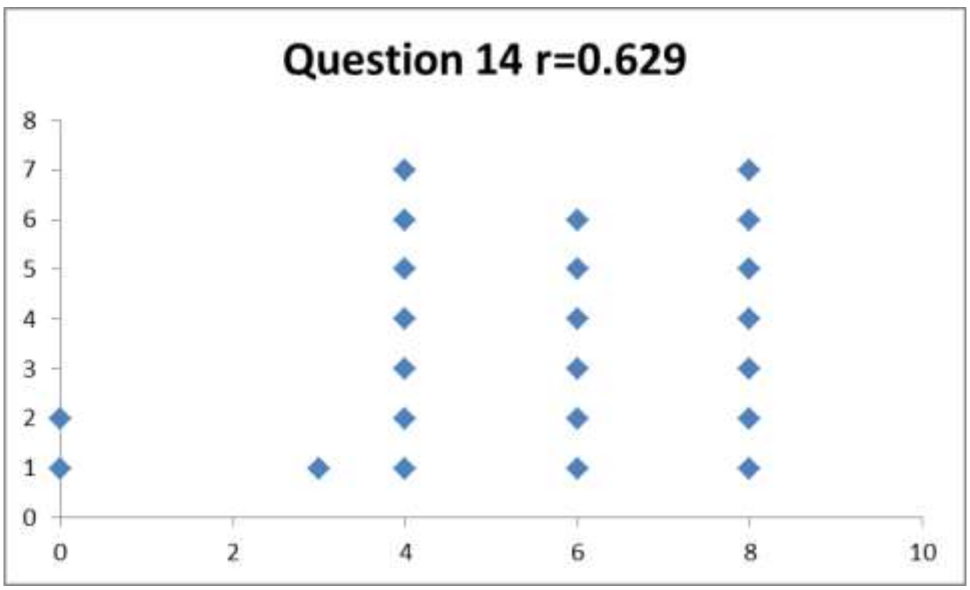


14) Let B_1 and B_2 be defined below. Find the change of basis matrix $[I]_{B_1}^{B_2}$.

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right\}, \qquad B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(8 points)

$$[I]_{B_1}^{B_2} = [I]_S^{B_2}[I]_{B_1}^S = \left([I]_{B_2}^S\right)^{-1}[I]_{B_1}^S = \begin{bmatrix} 1 & 4 & 1 \\ 0 & -3 & 1 \\ 3 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 4 & 0 \\ 2 & 4 & 2 \\ 3 & 6 & 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{7}{8} & \frac{31}{16} \\ -\frac{1}{4} & -\frac{1}{8} & -\frac{9}{16} \\ \frac{5}{4} & \frac{29}{8} & \frac{5}{16} \end{bmatrix}$$

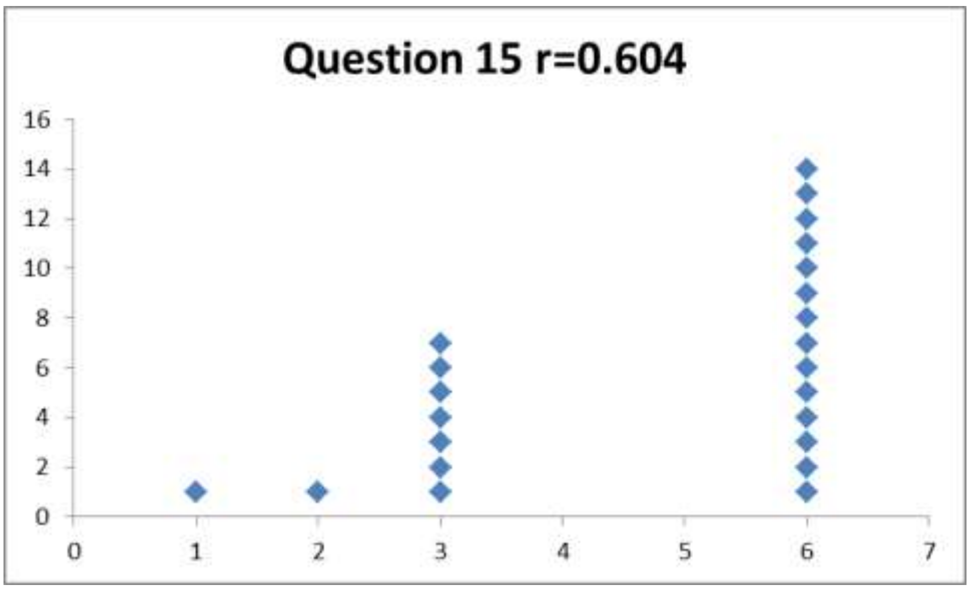


15) Define the linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$ via $T(\vec{x}) = A\vec{x}$. What is $T([1 \quad 2 \quad 4 \quad 2]^T)$?

$$A = \begin{bmatrix} 6 & 1 & 2 & 1 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

(6 points)

$$\begin{bmatrix} 6 & 1 & 2 & 1 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 18 \\ 5 \\ 14 \end{bmatrix}$$



16) Define the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^5$ via $T(\vec{x}) = A\vec{x}$ where A is defined below. Is T one-to-one? Why or why not?

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

(6 points)

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \\ 1 & 1 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We do not have a pivot in every column, meaning there are free variables, meaning it is not one-to-one!