\Name ____ Test 3, Fall 2017

The entire test is non-calculator. If a question says to "find a formula for…" you do not need to simplify anything or do any arithmetic. If it can be plugged into Wolfram Alpha using commands we've used in class, your answer is good enough!

Choose EIGHT of the first 10 problems to complete for 10 points each. Problem 11 cannot be skipped. Cross out the two problems you're skipping. Failure to do so will result in a -8 point penalty per question.

1) Let T be defined as below. Find a formula for $T^{-1}\left(\begin{bmatrix}7\\8\\9\end{bmatrix}\right)$.

$$T: \mathbb{R}^3 \to \mathbb{R}^3$$
$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \mapsto \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\0 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$T^{-1}\left(\begin{bmatrix}7\\8\\9\end{bmatrix}\right) = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\0 & 2 & 0\end{bmatrix}^{-1}\begin{bmatrix}7\\8\\9\end{bmatrix}$$

2) The linear transformations T is defined as below. Note that V has basis $\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix}\right\}$.

Find a formula for $\left[T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)\right]_S$

$$T: \mathbb{R}^3 \to V$$
$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \mapsto \begin{bmatrix}1 & 4 & 3\\4 & 5 & -2\\3 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$\begin{bmatrix}2 & 1 & -1\\0 & 3 & 1\\1 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 4 & 3\\4 & 5 & -2\\3 & 2 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

3) The linear transformation T is defined as below. Note that V has basis $\left\{\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$ and W has basis $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. Draw the diagram relating $V, W, \mathbb{R}^2$, and $\mathbb{R}^3$ then find a formula for $[T]_S^S$.

$$T: V \to W$$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto \begin{bmatrix}1 & 5\\2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 5\\2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}^{-1}$$

4) The linear transformations T and S are defined as below. Note that V has basis $\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$.

Find a formula for $[S \circ T]_S^S$.

$$T: \mathbb{R}^3 \to \mathbb{R}^2$$
$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \mapsto \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$S: \mathbb{R}^2 \to V$$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto \begin{bmatrix}1 & 2\\2 & 5\\2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\begin{bmatrix}2 & 1 & -1\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 5\\2 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$$

5) Given the matrix below, find the smaller of the two eigenspaces.

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$|xI - A| = \begin{vmatrix} x-1 & 0 & 0 \\ -4 & x-3 & -2 \\ -4 & -2 & x-3 \end{vmatrix} = (x-1)((x-3)^2 - 4) = (x-1)(x^2 - 6x + 5) = (x-1)^2(x-5)$$

Because $\lambda = 1$ is repeated, it could be a larger eigenspace. (Possibly dimension 2). Hence we'll look at $\lambda = 5$

$$\begin{bmatrix} 5-1 & 0 & 0 \\ -4 & 5-3 & -2 \\ -4 & -2 & 5-3 \end{bmatrix} \sim_R \begin{bmatrix} 4 & 0 & 0 \\ -4 & 2 & -2 \\ -4 & -2 & 2 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenspace is the null space of this, which is:

$$\text{span}\left(\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}\right)$$

6) Apply the Gram-Schmidt Process to find an orthonormal basis for S.

$$S = \text{span}\left(\left\{\begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}3\\4\\3\end{bmatrix}, \begin{bmatrix}4\\5\\10\end{bmatrix}\right\}\right)$$

Normalize the first vector: $\begin{bmatrix}2\\0\\0\end{bmatrix} \rightarrow \begin{bmatrix}1\\0\\0\end{bmatrix}$

$\vec{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

Remove old information from the second vector: $\begin{bmatrix}3\\4\\3\end{bmatrix} - \left\langle \begin{bmatrix}3\\4\\3\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\rangle \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\4\\3\end{bmatrix}$

Normalize the second vector: $\begin{bmatrix}0\\4\\3\end{bmatrix} \rightarrow \begin{bmatrix}0\\4/5\\3/5\end{bmatrix}$

$\vec{u}_2 = \begin{bmatrix}0\\4/5\\3/5\end{bmatrix}$

Remove old information from the third vector: $\begin{bmatrix}4\\5\\10\end{bmatrix} - \left\langle \begin{bmatrix}4\\5\\10\end{bmatrix}, \begin{bmatrix}0\\4/5\\3/5\end{bmatrix} \right\rangle \begin{bmatrix}0\\4/5\\3/5\end{bmatrix} - \left\langle \begin{bmatrix}4\\5\\10\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\rangle \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\-3\\4\end{bmatrix}$

Normalize the third vector: $\begin{bmatrix}0\\-3\\4\end{bmatrix} \rightarrow \begin{bmatrix}0\\-3/5\\4/5\end{bmatrix}$

$\vec{u}_3 = \begin{bmatrix}0\\-3/5\\4/5\end{bmatrix}$

We now have our orthonormal basis: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\4/5\\3/5\end{bmatrix}, \begin{bmatrix}0\\-3/5\\4/5\end{bmatrix}\right\}$

7) Diagonalize $\begin{bmatrix} 7 & -10 \\ 2 & -2 \end{bmatrix}$. Make sure your answer is in the form "$A = PDP^{-1}$".

We need a basis of $\mathbb{R}^2$ consisting of eigenvectors:

$$|xI - A| = \begin{vmatrix} x-7 & 10 \\ -2 & x+2 \end{vmatrix} = (x-7)(x+2) + 20 = x^2 - 5x + 6 = (x-2)(x-3)$$

$\lambda = 2$:
$\begin{bmatrix} 2-7 & 10 \\ -2 & 2+2 \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ -2 & 4 \end{bmatrix} \sim_R \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 3$:
$\begin{bmatrix} 3-7 & 10 \\ -2 & 3+2 \end{bmatrix} = \begin{bmatrix} -4 & 10 \\ -2 & 5 \end{bmatrix} \sim_R \begin{bmatrix} -2 & 5 \\ 0 & 0 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

The diagonalization is then:

$$\begin{bmatrix} 7 & -10 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}^{-1}$$

8) Suppose $T: \mathbb{R}^5 \rightarrow \mathbb{R}^7$ is one-to-one. How many solutions does $[T]\vec{x} = \vec{b}$ have? Justify your answer.

Because T is one-to-one, the corresponding system of equations can have no free variables. So it has either 0 or 1 solution depending on whether or not it is solvable at all.

Also note that linear equations over $\mathbb{R}$ always have either 0, 1, or infinitely many solutions. So any answer such as 2, 3, 4, 5, etc is obviously incorrect.

9) Suppose a 5×5 matrix A has determinant 2. How many solutions does the system below have? $[A|\vec{v}]\vec{x} = \vec{b}$. Here $\vec{v} = [1 \quad 2 \quad 3 \quad 4 \quad 5]^T$ and $\vec{b} = [1 \quad 2 \quad 3 \quad 4 \quad 6]^T$

Because $|A| = 2$, we know that every row of A has a pivot. Hence $[A|\vec{v}] = \vec{b}$ is solvable. Because of the 6th column, it has a free variable. Hence:

This has infinitely many solutions

10) Find the eigenvalue λ so that $\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$, then write down a formula for a and b.

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -1 + a + 3b \\ -2a + b \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda a \\ \lambda b \end{bmatrix}$$

$$\therefore \lambda = 3$$

$$-1 + a + 3b = 3a$$
$$-2a + b = 3b$$

$$-2a + 3b = 1$$
$$-2a - 2b = 0$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

11) Let A be an $n \times n$ matrix. You know that $\lambda = 0$ is an eigenvalue. What else can you say?

(4 points per insightful statement; 1 point per obvious statement; every incorrect statement nullifies a correct statement. 20 points maximum)

The fact that there is more than one solution means that we have the negation of everything in the big theorem:

- A is not invertible.
- A is singular.
- $A\vec{x} = \vec{0}$ has more than the trivial solution. (Can't use this one – it was what was given!!!)
- A is not a product of elementary matrices.
- Not every row has a pivot. (In reduced row echelon form)
- Not every column has a pivot. (In reduced row echelon form)
- $rank(A) < n$
- The rows of A are linearly dependent
- $RS(A) \subset \mathbb{R}^n$ (proper subset)
- $\dim(RS(A)) < n$
- The columns of A are linearly dependent
- $CS(A) \subset \mathbb{R}^n$ (proper subset)
- $\dim(CS(A)) < n$
- $|A| = 0$
- T does not has an inverse.
- $[T]$ is singular
- T is not one-to-one
- T is not onto

Color scheme:

Green ink – your statement doesn't make sense. I don't mean it's wrong or right, it's impossible to evaluate whether it's right or wrong because the grammar literally doesn't make sense. Think about it as a compile error in a computer program: we can't evaluate if the logic is right or wrong because there isn't any output.

If the grammar is close enough to correct that I can make sense of it, half credit was awarded. Denoted with a circled "2".

Blue ink – incorrect statements that cancelled a correct statement.

Pink ink – Something can be correctly said, but more information is needed. Likely an ambiguous pronoun, such as "it has infinitely many solutions". What is "it"?

Orange ink – Things that are just too obvious to give any points for. Such as "A nas n columns." Yeah. I told you that!

Half credit was given for a valiant effort of things that are all the negation of the correct answers. (max 10 points)