Name ____ Test 3, Fall 2019

1) A linear transformation is given by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 3x_2 \\ x_3 \end{bmatrix}$. Find the kernel of T. (10 points)

First find $[T] = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then pull out the two equations $x_1 - 3x_2 = 0$ and $x_3 = 0$. We then see that a everything in the null space is given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix}$$

Use a basis vector here, say with $x_2 = 1$, to get a the whole vector space:

$$span\left(\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}\right\}\right)$$

This is the kernel of the linear transformation.

2) Determine whether or not the linear transformation from the previous problem is onto and justify your answer. (10 points)

It is indeed onto – if you look at the matrix representation you see that there is a pivot in each row.

3) Suppose a linear transformation satisfies the equations below. Find $[T]$. (10 points)

$$T\left(\begin{bmatrix}2\\0\end{bmatrix}\right) = \begin{bmatrix}2\\6\end{bmatrix} \quad T\left(\begin{bmatrix}0\\3\end{bmatrix}\right) = \begin{bmatrix}6\\12\end{bmatrix}$$

First let's scale each of these down to the standard basis vectors:

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\end{bmatrix} \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\4\end{bmatrix}$$

Now we can construct the matrix itself:

$$[T] = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$$

4) Let $B_1 = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}\right\}$ and $B_2 = \left\{\begin{bmatrix}5\\4\end{bmatrix}, \begin{bmatrix}0\\6\end{bmatrix}\right\}$. Find a formula for $[I]_{B_1}^{B_2}$, the change of basis matrix from B_1 to B_2. (10 points)

From the diagram below, we see that $[I]_{B_1}^{B_2} = \begin{bmatrix}5 & 0\\4 & 6\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}$

$$\begin{tikzcd}
\mathbb{R}^2_{B_1} \arrow[rr] \arrow[dr, "\begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}"'] & & \mathbb{R}^2_{B_2} \arrow[dl, "\begin{bmatrix}5 & 0\\4 & 6\end{bmatrix}"] \\
& \mathbb{R}^2_S &
\end{tikzcd}$$

5) Let B_1 and B_2 both be bases for $\mathbb{R}^3$. Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ with matrix representation A under the standard basis. Find a formula for $[A^{-1}]_{B_1}^{B_2}$. (10 points)

Whoops; it should have said $[A^{-1}]_{B_2}^{B_1}$ in which case we get the answer below. Full credit for everyone, because the question makes no sense as stated. A^{-1} must start in B_2 and end in B_1.

$$[A^{-1}]_{B_2}^{B_1} = \left([I]_{B_1}^{S}\right)^{-1} A^{-1} [I]_{B_2}^{S}$$

$$\begin{tikzcd}
\mathbb{R}^3_{B_1} \arrow[r, "{[T]_{B_1}^{B_2}}"] \arrow[d, "{[I]_{B_1}^{S}}"'] & \mathbb{R}^3_{B_2} \arrow[d, "{[I]_{B_2}^{S}}"] \\
\mathbb{R}^3_{S} \arrow[r, "{[T]_S^S = A}"] & \mathbb{R}^3_{S}
\end{tikzcd}$$

6) Find a diagonalization of the matrix below. (10 points)

$$\begin{bmatrix} 7 & -3 \\ 6 & -2 \end{bmatrix}$$

First find the eigenvalues:

$$\begin{vmatrix} x-7 & 3 \\ -6 & x+2 \end{vmatrix} = (x-7)(x+2)+18 = x^2-5x-14+18 = x^2-5x+4 = (x-4)(x-1)$$

Eigenvalue $\lambda = 1$:

$$\begin{bmatrix} 1-7 & 3 \\ -6 & 1+2 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ -6 & 3 \end{bmatrix} \sim_R \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

Eigenvector: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Eigenvalue $\lambda = 4$:

$$\begin{bmatrix} 4-7 & 3 \\ -6 & 4+2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -6 & 6 \end{bmatrix} \sim_R \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Eigenvector: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Diagonalization:

$$\begin{bmatrix} 7 & -3 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

7) Below a basis is given. Find an orthogonal version of this basis. (10 points)

$$\left\{\begin{bmatrix}2\\0\\-2\end{bmatrix},\ \begin{bmatrix}1\\1\\2\end{bmatrix}\right\}$$

$$\vec{b}_1 = \begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

$$proj_{\vec{b}_1}(\vec{v}_2) = \frac{2-4}{2^2+2^2}\begin{bmatrix}2\\0\\-2\end{bmatrix} = \frac{-2}{8}\begin{bmatrix}2\\0\\-2\end{bmatrix} = \begin{bmatrix}-1/2\\0\\1/2\end{bmatrix}$$

$$\vec{b}_2 = \begin{bmatrix}1\\1\\2\end{bmatrix} - \begin{bmatrix}-1/2\\0\\1/2\end{bmatrix} = \begin{bmatrix}3/2\\1\\3/2\end{bmatrix}$$

Answer: $\left\{\begin{bmatrix}2\\0\\-2\end{bmatrix}, \begin{bmatrix}3/2\\1\\3/2\end{bmatrix}\right\}$

8) Suppose that a 8×8 matrix is invertible. What is something else that can be said about the corresponding system of equations $A\vec{x} = \vec{0}$? (5 points)

There are multiple answers, such as: it has only the zero solution.

Make sure you talk about the system of equation $A\vec{x} = \vec{0}$ and not the matrix A nor corresponding linear transformation T.

9) Suppose that a 8×8 matrix is invertible. What is something else that can be said about the corresponding linear transformation T? (5 points)

There are multiple answers, such as: It is invertible.

Make sure you talk about the linear transformation T and not the corresponding matrix A or systems of equations.

10) Suppose that a 8×8 matrix is invertible. What is something else that can be said about the matrix itself? (5 points)

There are multiple answers, such as: it has rank 8.

Make sure you talk about the matrix and not the corresponding linear transformation T nor systems of equations.

11) Suppose that a linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^4$ is onto. What is the maximum dimension of the solution set to $A\vec{x} = \vec{0}$, where A is the matrix representation of T. (5 points)

Dimension 0, the only solution is $\vec{x} = \vec{0}$, which gives rise to the 0-dimensional space $\{\vec{0}\}$.

12) Suppose that a linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^6$ is one-to-one. What is the maximum dimension of the solution set to $A\vec{x} = \vec{0}$, where A is the matrix representation of T. (5 points)

Dimension 0, the only solution is $\vec{x} = \vec{0}$, which gives rise to the 0-dimensional space $\{\vec{0}\}$.

13) Suppose that a linear transformation $T: \mathbb{R}^7 \to \mathbb{R}^7$ is invertible. What is the minimum rank of A, where A is the matrix representation of T. (5 points)

Rank 7, exactly. So the minimum rank is 7.