Name ____ Test 1, Spring 2019

1) Multiply the matrices below, or state why it is impossible.
(+1/-3 points)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 23 \\ 2 & 9 \end{bmatrix}$$

2) Given the system of equations below, write the corresponding matrix equation $A\vec{x} = \vec{b}$.
(+2/-3 points)

$$2x_1 + 8x_2 - 4x_3 = -10$$
$$-x_1 - 3x_2 + 5x_3 = 4$$

$$\begin{bmatrix} 2 & 8 & -4 \\ -1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \end{bmatrix}$$

3) Assume B and D are 3×2 matrices, while C is a 2×3 matrix. Simplify the expression below or explain why it is not possible.

(+2/-3 points)

$$B(C^T + D)$$

This is not possible because the sizes do not match up. In particular B is 3×2, but $C^T + D$ is 3×2. We cannot multiply a matrix with 2 columns by a matrix with 3 rows.

4) Compute $\vec{u} - \vec{v}$, given the definitions below.

(+1/-5 points)

$$\vec{u} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}; \vec{v} = \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix}$$

$$\vec{u} - \vec{v} = \begin{bmatrix} 7 \\ -3 \\ -5 \end{bmatrix}$$

5) Given the definitions below, express $\vec{b}$ as a linear combination of $\vec{a}_1$ and $\vec{a}_2$, or explain why it is not possible.
(+3/-1 points)

$$\vec{a}_1 = \begin{bmatrix} 4 \\ -6 \end{bmatrix}; \vec{a}_2 = \begin{bmatrix} -6 \\ 9 \end{bmatrix}; \vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

This is not possible because as we see below, the augmented matrix corresponds to an inconsistent system of linear equations.

$$\begin{bmatrix} 4 & -6 & 1 \\ -6 & 9 & 5 \end{bmatrix} \sim_R \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{4} \\ -6 & 9 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{4} \\ 0 & 0 & \frac{13}{2} \end{bmatrix}$$

OR note that $\vec{a}_2$ is a multiple of $\vec{a}_1$ so all linear combinations of these are just multiples of $\vec{a}_1$, and that $\vec{b}$ is not such a multiple.

6) Row reduce the matrix below.

(+2/-2 points)

$$\begin{bmatrix} -2 & -4 & 0 \\ 6 & 12 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -4 & 0 \\ 6 & 12 & 7 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 2 & 0 \\ 6 & 12 & 7 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7) Solve the system of equations below.

(Hint: look at the previous question)

(+2/-3 points)

$$-2x_1 - 4x_2 = 0$$
$$6x_1 + 12x_2 = 7$$

From the row reduced version, we see that this has no solutions.

8) Are the vectors $\begin{bmatrix}-2\\6\end{bmatrix}$, $\begin{bmatrix}-4\\12\end{bmatrix}$, and $\begin{bmatrix}0\\7\end{bmatrix}$ linearly independent? Justify your answer.

(+3/-2 points)

No, because the matrix $\begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\end{bmatrix}$ does not have a pivot in every column.

True or False: circle the correct answer. A statement is true if it is *always* true; false if it is *ever* false.

T or F 9) The set $\mathbb{R}^{m\times n}$ together with the usual definition of matrix addition and scalar multiplication is a vector space.

T or F 10) Four distinct vectors in $\mathbb{R}^3$ will span $\mathbb{R}^3$.

T or F 11) Three distinct nonzero vectors in $\mathbb{R}^4$ will be linearly independent.

T or F 12) The set of vectors $\left\{\begin{bmatrix}a\\b\end{bmatrix} : a, b \in \mathbb{Q}\right\}$ is a subspace of $\mathbb{R}^2$.

(+1/-1 point each)

Note: The pink re-grading came because different versions had the problems listed in a different order. I forgot about that when grading the first time …

13) A square matrix A is called <u>idempotent</u> if $A^2 = A$. Find three examples of 2×2 matrices that are idempotent.

(+3/-1 points)

Two examples are really easy to get our hands on:

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ & $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A third takes a bit of cleverness:

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

There are many answers.

14) Find three vectors that are in the span of $\begin{bmatrix}2\\6\end{bmatrix}$ and $\begin{bmatrix}9\\15\end{bmatrix}$.

(+3/-2 points)

Any linear combination of these two things. Obvious choices are $\begin{bmatrix}2\\6\end{bmatrix}$, $\begin{bmatrix}9\\15\end{bmatrix}$, and $\begin{bmatrix}0\\0\end{bmatrix}$. However, if we note that these two vectors are linearly independent, actually any vector in $\mathbb{R}^2$ is in their span.

15) Find a vector that is not in the span of $\begin{bmatrix}1\\3\\-2\end{bmatrix}$ and $\begin{bmatrix}0\\-1\\1\end{bmatrix}$

(+3/-2 points)

We must find something that is not a linear combination of these two vectors. You can either do this via row reduction or inspection.

Through row reduction:

$$\begin{bmatrix}1 & 0 & a\\3 & -1 & b\\-2 & 1 & c\end{bmatrix} \sim_R \begin{bmatrix}1 & 0 & a\\0 & -1 & b-3a\\0 & 1 & c+2a\end{bmatrix} \sim_R \begin{bmatrix}1 & 0 & a\\0 & 1 & 3a-b\\0 & 1 & c+2a\end{bmatrix} \sim_R \begin{bmatrix}1 & 0 & a\\0 & 1 & 3a-b\\0 & 0 & c+b-a\end{bmatrix}$$

Things not in the span will have a third pivot, meaning $c+b-a \neq 0$. In order to be in the span, we need a free variable: $c+b-a=0$. How about $a=b=0$ and $c=1$:

$$\begin{bmatrix}0\\0\\1\end{bmatrix}$$

OR though inspection:

How about $\begin{bmatrix}1\\2\\3\end{bmatrix}$? Based on the x and z coordinate, let's see what vector has $\begin{bmatrix}1\\?\\3\end{bmatrix}$:

$\begin{bmatrix}1\\3\\-2\end{bmatrix} + 5\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\-2\\3\end{bmatrix}$, so our choice was indeed not in the span.

For the problems on this page, use the matrix below on the left. It's row reduced echelon form is shown on the right.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & -1 & 4 \\ 1 & 4 & -2 & 4 \\ 2 & 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

16) Find the row space of A, $RS(A)$.
(+2/-3 points)

$$\text{span}(\{[1 \quad 2 \quad 0 \quad -1], [-2 \quad -3 \quad -1 \quad 4], [1 \quad 4 \quad -2 \quad 4], [2 \quad 2 \quad 2 \quad -4]\})$$

OR

$$span(\{[1 \quad 0 \quad 2 \quad 0], [0 \quad 1 \quad -1 \quad 0], [0 \quad 0 \quad 0 \quad 1]\})$$

17) Find the column space of A, $CS(A)$.
(+3/-2 points)

$$span\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 4 \\ -4 \end{bmatrix}\right\}\right)$$

OR

$$span\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 4 \\ -4 \end{bmatrix}\right\}\right)$$

18) Find the null space of $A, NS(A)$.

(+4/-2 points)

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third row tells us: $x_4 = 0$

The second row tells us: $x_2 - x_3 = 0$, so $x_2 = x_3$

The first row tells us: $x_1 + 2x_3 = 0$, so $x_1 = -2x_3$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore NS(A) = span\left(\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

For the problems on this page, use the matrix below on the left. Its row reduced echelon form is shown on the right.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & -1 & 4 \\ 1 & 4 & -2 & 4 \\ 2 & 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

19) Determine if the homogeneous system $A\vec{x} = \vec{0}$ has any nontrivial solutions.

(+4/-3 points)

Yes, because we see that the third column corresponds to a free variable.

20) Find a basis for the span of the columns of A.

(+2/-2 points)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 4 \\ -4 \end{bmatrix} \right\}$$

21) Expand the set $\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}\right\}$ to a basis of $\mathbb{R}^3$

(+2/-3 points)

$$\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

22) Let A be an 7×5 matrix. You know that $A\vec{x} = \vec{b}$ has no solutions. What else can you say?
(Maximum +4/-4 points) (You may list as many statements as you like, each insightful statement is worth +2 or -2 points.)

Here is a selection of statements that could be true, or false, as well as things we do not know or do not apply.

Statements on solutions and free variables:

The equation $A\vec{x} = \vec{b}$ has a free variable – UNKNOWN

The equation $A\vec{x} = \vec{0}$ has infinitely many solutions – UNKNOWN

The equation $A\vec{x} = \vec{0}$ has a solution – TRIVIALLY TRUE

Statements on linear dependence and independence of rows and columns:

The rows of A are linearly dependent – TRUE

The rows of A are linearly independent – FALSE

The columns of A are linearly independent – UNKNOWN

The columns of A are linearly dependent – UNKNOWN

Statements on the matrix itself

Every row has a pivot – FALSE

Every column has a pivot – UNKNOWN

In echelon form A has a row of zeroes – TRUE

The row space spans $\mathbb{R}^5$ – UNKNOWN

The column space spans $\mathbb{R}^7$ – FALSE

The null space is nontrivial – UNKNOWN

The matrix is invertible – DOES NOT APPLY TO NONSQUARE MATRICES

The matrix is a product of elementary matrices – DOES NOT APPLY TO NONSQUARE MATRICES

The matrix is singular – DOES NOT APPLY TO NONSQUARE MATRICES