The entire test is non-calculator. If a question says to “find a formula for...” you do not need to simplify anything or do any arithmetic. If it can be plugged into a computer algebra system such as Wolfram Alpha using commands we’ve used in class, your answer is good enough!

Choose EIGHT of the first 10 problems to complete for (+5/-5) points each. Problem 11 cannot be skipped. If you have extra time and want to gamble for extra credit, you can attempt additional problems: for each additional problem attempted you will receive a -2 point penalty along with whatever point(s) you earn on that problem.

1) Let $V$ be the set of polynomials of degree at most 2 with basis $B = \{x^2, x, 1\}$. Also define a linear transformation $T: V \to V$ via $T(ax^2 + bx + c) = 3a - 2cx^2$. Find the matrix representation of $T$, $[T]_B^B$. 
2) Let $B_1$ and $B_2$ be bases of $\mathbb{R}^5$ and $\mathbb{R}^7$ respectively. A linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^7$ is also defined. Draw the diagram that illustrates the interplay between the following eight mathematical objects:

$\mathbb{R}^5$, $\mathbb{R}_{B_1}^5$, $\mathbb{R}_S^5$, $\mathbb{R}_{B_2}^7$, $[T]_{B_1}^{B_2}$, $[T]_S^S$, $[I]_{B_1}^S$, $[I]_{B_2}^S$
3) Given the information below, find a formula for the change of basis matrix $[I]_{B_2}^{B_1}$

\[ B_1 = \{(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}) \} \quad B_2 = \{(\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \end{bmatrix}) \} \]
4) Let $U$ and $V$ be linear transformations as defined below. Find a formula for the matrix representing $[V \circ U]_S$.

Let $U : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $V : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined as:

- $U : \mathbb{R}^3 \rightarrow \mathbb{R}^3$
  
  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix}$

- $V : \mathbb{R}^3 \rightarrow \mathbb{R}^3$
  
  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 5x + 5y + 6z \\ 8x + 8y + 7z \\ 9x + 10y + z \end{bmatrix}$
5) Given the information below, find a formula for $[T^{-1}]_{B_2}^S$.

$$B_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{5}{6} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_S\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}_S$$
6) Find an eigenvector of the matrix below. Show your work.

\[
\begin{bmatrix}
2 & 4 & -4 \\
0 & 1 & 1 \\
0 & -3 & 5
\end{bmatrix}
\]
7) The matrix $A$ has eigenvalue $\lambda_1 = 2$ with eigenvector \[
\begin{bmatrix}
2 \\
0 \\
1
\end{bmatrix},
\] eigenvalue $-7$ with eigenvectors \[
\begin{bmatrix}
1 \\
3 \\
0
\end{bmatrix}
\] and \[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]
Give its diagonalization.
8) Given the basis \( \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\} \) of a vector space, find an orthogonal basis spanning the same vector space.
9) Let $V$ be the vector space $\mathbb{R}^3$ with basis $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Given the linear transformation below, find a formula for $T \left( \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix} \right)$.

$T: V \rightarrow \mathbb{R}^3$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix}$
10) Let $V$ be the vector space $\mathbb{R}^3$ with basis $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$.

Given the coordinate vector $[\vec{v}]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_B$, find $[\vec{v}]_S$. 
11) Let $A$ be an $3 \times 3$ matrix. You know that it has eigenvalues $\lambda_1 = 2, \lambda_2 = 5, \lambda_3 = 10$. What else can you say?
(Maximum +10/-10 points) (You may list as many statements as you like, each insightful statement is worth +2 or -2 points.)