Name ____ Test 3, Fall 2019

The entire test is non-calculator. If a question says to "find a formula for…" you do not need to simplify anything or do any arithmetic. If it can be plugged into a computer algebra system such as Wolfram Alpha using commands we've used in class, your answer is good enough!

Choose EIGHT of the first 10 problems to complete for (+5/-5) points each. Problem 11 cannot be skipped. If you have extra time and want to gamble for extra credit, you can attempt additional problems: for each additional problem attempted you will receive a -2 point penalty along with whatever point(s) you earn on that problem.

1) Let V be the set of polynomials of degree at most 2 with basis $B = \{x^2, x, 1\}$. Also define a linear transformation $T: V \to V$ via $T(ax^2 + bx + c) = 3a - 2cx^2$. Find the matrix representation of T, $[T]_B^B$.

2) Let B_1 and B_2 be bases of $\mathbb{R}^5$ and $\mathbb{R}^7$ respectively. A linear transformation $T: \mathbb{R}^5 \rightarrow \mathbb{R}^7$ is also defined. Draw the diagram that illustrates the interplay between the following eight mathematical objects:

$$\mathbb{R}^5_S,\ \mathbb{R}^5_{B_1},\ \mathbb{R}^7_S,\ \mathbb{R}^7_{B_2},\ [T]^{B_2}_{B_1},\ [T]^S_S,\ [I]^S_{B_1},\ [I]^S_{B_2}$$

3) Given the information below, find a formula for the change of basis matrix $[I]_{B_1}^{B_2}$

$$B_1 = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\right\} \quad B_2 = \left\{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\9\end{bmatrix}\right\}$$

4) Let U and V be linear transformations as defined below. Find a formula for the matrix representing $[V \circ U]_S$.

$$U: \mathbb{R}^3 \to \mathbb{R}^3 \qquad V: \mathbb{R}^3 \to \mathbb{R}^3$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 5x + 5y + 6z \\ 8x + 8y + 7z \\ 9x + 10y + z \end{bmatrix}$$

5) Given the information below, find a formula for $[T^{-1}]_{B_2}^{S}$.

$$B_1 = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\right\}\ B_2 = \left\{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\9\end{bmatrix}\right\}, T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}_S\right) = \begin{bmatrix}x_1 + x_2\\x_2\end{bmatrix}_S$$

6) Find an eigenvector of the matrix below. Show your work.

$$\begin{bmatrix} 2 & 4 & -4 \\ 0 & 1 & 1 \\ 0 & -3 & 5 \end{bmatrix}$$

7) The matrix A has eigenvalue $\lambda_1 = 2$ with eigenvector $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, eigenvalue -7 with eigenvectors $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Give its diagonalization.

8) Given the basis $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}$ of a vector space, find an orthogonal basis spanning the same vector space.

9) Let V be the vector space $\mathbb{R}^3$ with basis $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$. Given the linear transformation below, find a formula for $T\left(\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}_B \right)$.

$$T: V \to \mathbb{R}^3_S$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix}$$

10) Let V be the vector space $\mathbb{R}^3$ with basis $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$.

Given the coordinate vector $[\vec{v}]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_B$, find $[\vec{v}]_S$.

11) Let A be an 3×3 matrix. You know that it has eigenvalues $\lambda_1 = 2, \lambda_2 = 5, \lambda_3 = 10$. What else can you say?

(Maximum +10/-10 points) (You may list as many statements as you like, each insightful statement is worth +2 or -2 points.)