The entire test is non-calculator. If a question says to “find a formula for...” you do not need to simplify anything or do any arithmetic. If it can be plugged into a computer algebra system such as Wolfram Alpha using commands we’ve used in class, your answer is good enough!

Choose EIGHT of the first 10 problems to complete for (+5/-5) points each. Problem 11 cannot be skipped. If you have extra time and want to gamble for extra credit, you can attempt additional problems: for each additional problem attempted you will receive a -2 point penalty along with whatever point(s) you earn on that problem.

1) Let $V$ be the set of polynomials of degree at most 2 with basis $B = \{ x^2, x, 1 \}$. Also define a linear transformation $T: V \rightarrow V$ via $T(\alpha x^2 + \beta x + \gamma) = 3\alpha - 2c x^2$. Find the matrix representation of $T$, $[T]_B^B$.

$$
\begin{bmatrix}
0 & 0 & -2 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
$$
2) Let \( B_1 \) and \( B_2 \) be bases of \( \mathbb{R}^5 \) and \( \mathbb{R}^7 \) respectively. A linear transformation \( T : \mathbb{R}^5 \rightarrow \mathbb{R}^7 \) is also defined. Draw the diagram that illustrates the interplay between the following eight mathematical objects:

\[
\mathbb{R}^5_B, \mathbb{R}^5_S, \mathbb{R}^7_B, \mathbb{R}^7_S, [T]_{B_1}^{B_2}, [T]_S^S, [I]_{B_1}^S, [I]_{B_2}^S
\]
3) Given the information below, find a formula for the change of basis matrix \([I]_{B_1}^{B_2}\)

\[B_1 = \{\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}\} \quad B_2 = \{\begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \end{bmatrix}\}\]

\[\begin{bmatrix} [I]_{B_2}^{B_1} = [I]_{S}^{B_2} [I]_{S}^{B_1} = ([I]_{S}^{B_2})^{-1} [I]_{S}^{B_1} = \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}\end{bmatrix}\]
4) Let $U$ and $V$ be linear transformations as defined below. Find a formula for the matrix representing $[V \circ U]_S$.

$U : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$[x, y, z] \mapsto [x + 2y + 3z, 4x + 5y + 6z, 7x + 8y + 9z]$

$V : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$[x, y, z] \mapsto [5x + 5y + 6z, 8x + 8y + 7z, 9x + 10y + z]$

$[5 \ 5 \ 6 \ 1 \ 2 \ 3]$

$[8 \ 8 \ 7 \ 4 \ 5 \ 6]$

$[9 \ 10 \ 1 \ 7 \ 8 \ 9]$
5) Given the information below, find a formula for $[T^{-1}]_{B_2}^S$.

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\} \quad B_2 = \left\{ \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \end{bmatrix} \right\}, T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}_S$$

$$[T]^S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$([T]^S)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$[T^{-1}]_{B_2}^S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix}$$
6) Find an eigenvector of the matrix below. Show your work.

\[
\begin{bmatrix}
2 & 4 & -4 \\
0 & 1 & 1 \\
0 & -3 & 5
\end{bmatrix}
\]

\[
|A - xI| = \begin{vmatrix} 2 - x & 4 & -4 \\ 0 & 1 - x & 1 \\ 0 & -3 & 5 - x \end{vmatrix} = (2 - x)((1 - x)(5 - x) + 3) = (2 - x)(x^2 - 6x + 5 + 3)
\]

\[
= (2 - x)(x^2 - 6x + 8) = (2 - x)(x - 2)(x - 4) = -(x - 2)^2(x - 4)
\]

The eigenvalues are 2 and 4. We could have seen \( \lambda = 2 \) without doing so much work, however. So let’s use that eigenvalue:

\[
A - 2I = \begin{bmatrix}
0 & 4 & -4 \\
0 & -1 & 1 \\
0 & -3 & 3
\end{bmatrix} \sim_r \begin{bmatrix}
0 & 1 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

From the above we see that this eigenspace is: \( \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \). Choosing one vector in this space, \( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \), we get our answer:

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]
7) The matrix $A$ has eigenvalue $\lambda_1 = 2$ with eigenvector $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, eigenvalue $-7$ with eigenvectors $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Give its diagonalization.

$$
\begin{bmatrix}
2 & 1 & 1 \\
0 & 3 & 1 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 2 & 0 & 0 \\ 0 & -7 & 0 \\ 1 & 0 & -7 \end{bmatrix}
\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1}
$$
8) Given the basis \( \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\} \) of a vector space, find an orthogonal basis spanning the same vector space.

\( \vec{b}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \)

\( \vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \)

\( \text{proj}_{\vec{v}_1}(\vec{v}_1) = \frac{2}{5} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ 0 \\ \frac{2}{5} \end{pmatrix} \)

\( \vec{b}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{4}{5} \\ 0 \\ \frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ 3 \\ -\frac{2}{5} \end{pmatrix} \)
9) Let $V$ be the vector space $\mathbb{R}^3$ with basis $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$. Given the linear transformation below, find a formula for $T \left( \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}_B \right)$.

$$T: V \rightarrow \mathbb{R}^3$$
$$[x, y, z] \mapsto [x + 2y + 3z, 4x + 5y + 6z, 7x + 8y + 9z]$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$$

**Note that the actual definition of $B$ didn’t affect our answer, because $T$ was encoded on $V$ itself.**
10) Let $V$ be the vector space $\mathbb{R}^3$ with basis $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$.

Given the coordinate vector $[\vec{v}]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_B$, find $[\vec{v}]_S$.

\[
\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}
\]
11) Let $A$ be an $3 \times 3$ matrix. You know that it has eigenvalues $\lambda_1 = 2, \lambda_2 = 5, \lambda_3 = 10$. What else can you say?

(Maximum +10/-10 points) (You may list as many statements as you like, each insightful statement is worth +2 or -2 points.)

This is the big theorem. Here is a selection of statements that we can say. The key piece of information to notice is that $\lambda = 0$ is not an eigenvalue, as a $3 \times 3$ matrix can have only 3 eigenvalues. That means that $A$ is nonsingular. Other things we can say are:

**Statements on solutions and free variables:**
- The equation $A \vec{x} = \vec{b}$ has a free variable – FALSE
- The equation $A \vec{x} = \vec{b}$ has a unique solution for each $\vec{b}$ – TRUE
- The equation $A \vec{x} = \vec{0}$ has infinitely many solutions – FALSE
- The equation $A \vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as a solution - TRUE

**Statements on linear dependence and independence of rows and columns:**
- The rows of $A$ are linearly dependent – FALSE
- The rows of $A$ are linearly independent – TRUE
- The columns of $A$ are linearly independent – TRUE
- The columns of $A$ are linearly dependent – FALSE

**Statements on the matrix itself**
- In echelon form, every row has a pivot – TRUE
- In echelon form every column has a pivot – TRUE
- In echelon form $A$ has a row of zeroes – FALSE
- The row space spans $\mathbb{R}^3$ – TRUE
- The column space spans $\mathbb{R}^3$ – TRUE
- The null space is nontrivial – FALSE
- The matrix is invertible – TRUE
- The matrix is a product of elementary matrices – TRUE
- The matrix is singular – FALSE
- The rank of the matrix is 3 – TRUE
- The determinant of the matrix is zero – FALSE

**Statements on the corresponding linear transformation**
- The corresponding linear transformation is one-to-one – TRUE
- The corresponding linear transformation is onto – TRUE

**Statements on eigenvalues and diagonalization**
- $\lambda = 0$ is not an eigenvalue - TRUE
- The matrix is diagonalizable – TRUE