Name ____ Test 3, Fall 2019

The entire test is non-calculator. If a question says to "find a formula for…" you do not need to simplify anything or do any arithmetic. If it can be plugged into a computer algebra system such as Wolfram Alpha using commands we've used in class, your answer is good enough!

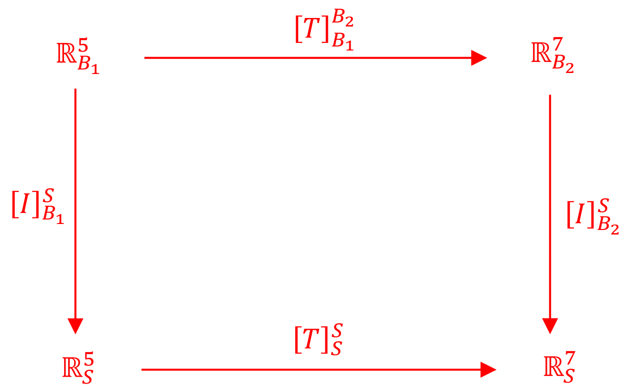
Choose EIGHT of the first 10 problems to complete for (+5/-5) points each. Problem 11 cannot be skipped. If you have extra time and want to gamble for extra credit, you can attempt additional problems: for each additional problem attempted you will receive a -2 point penalty along with whatever point(s) you earn on that problem.

1) Let V be the set of polynomials of degree at most 2 with basis $B = \{x^2, x, 1\}$. Also define a linear transformation $T: V \rightarrow V$ via $T(ax^2 + bx + c) = 3a - 2cx^2$. Find the matrix representation of T, $[T]_B^B$.

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

2) Let B_1 and B_2 be bases of $\mathbb{R}^5$ and $\mathbb{R}^7$ respectively. A linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^7$ is also defined. Draw the diagram that illustrates the interplay between the following eight mathematical objects:

$$\mathbb{R}^5_S,\ \mathbb{R}^5_{B_1},\ \mathbb{R}^7_S,\ \mathbb{R}^7_{B_2},\ [T]^{B_2}_{B_1},\ [T]^S_S,\ [I]^S_{B_1},\ [I]^S_{B_2}$$



3) Given the information below, find a formula for the change of basis matrix $[I]_{B_1}^{B_2}$

$$B_1 = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\right\} \quad B_2 = \left\{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\9\end{bmatrix}\right\}$$

$$[I]_{B_1}^{B_2} = [I]_S^{B_2}[I]_{B_1}^S = \left([I]_{B_2}^S\right)^{-1}[I]_{B_1}^S = \begin{bmatrix}3 & 0\\0 & 9\end{bmatrix}^{-1}\begin{bmatrix}1 & 5\\2 & 6\end{bmatrix}$$

4) Let U and V be linear transformations as defined below. Find a formula for the matrix representing $[V \circ U]_S$.

$$U: \mathbb{R}^3 \to \mathbb{R}^3 \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix}$$

$$V: \mathbb{R}^3 \to \mathbb{R}^3 \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 5x + 5y + 6z \\ 8x + 8y + 7z \\ 9x + 10y + z \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 & 6 \\ 8 & 8 & 7 \\ 9 & 10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

5) Given the information below, find a formula for $[T^{-1}]_{B_2}^{S}$.

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\} \; B_2 = \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \end{bmatrix} \right\}, T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_S \right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}_S$$

$$[T]_S^S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\left([T]_S^S\right)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$[T^{-1}]_{B_2}^{S} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix}$$

6) Find an eigenvector of the matrix below. Show your work.

$$\begin{bmatrix} 2 & 4 & -4 \\ 0 & 1 & 1 \\ 0 & -3 & 5 \end{bmatrix}$$

$$|A - xI| = \begin{vmatrix} 2-x & 4 & -4 \\ 0 & 1-x & 1 \\ 0 & -3 & 5-x \end{vmatrix} = (2-x)\big((1-x)(5-x)+3\big) = (2-x)(x^2 - 6x + 5 + 3)$$
$$= (2-x)(x^2 - 6x + 8) = (2-x)(x-2)(x-4) = -(x-2)^2(x-4)$$

The eigenvalues are 2 and 4. We could have seen $\lambda = 2$ without doing so much work, however. So let's use that eigenvalue:

$$A - 2I = \begin{bmatrix} 0 & 4 & -4 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \end{bmatrix} \sim_R \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the above we see that this eigenspace is: $span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$. Choosing one vector in this space, , we get our answer:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

7) The matrix A has eigenvalue $\lambda_1 = 2$ with eigenvector $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, eigenvalue -7 with eigenvectors $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Give its diagonalization.

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

8) Given the basis $\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\3\\0\end{bmatrix}\right\}$ of a vector space, find an orthogonal basis spanning the same vector space.

$$\vec{b}_1 = \begin{bmatrix}2\\0\\1\end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix}1\\3\\0\end{bmatrix}$$

$$proj_{\vec{b}_1}(\vec{v}_1) = \frac{2}{5}\begin{bmatrix}2\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{4}{5}\\0\\\frac{2}{5}\end{bmatrix}$$

$$\vec{b}_2 = \begin{bmatrix}1\\3\\0\end{bmatrix} - \begin{bmatrix}\frac{4}{5}\\0\\\frac{2}{5}\end{bmatrix} = \begin{bmatrix}\frac{1}{5}\\3\\-\frac{2}{5}\end{bmatrix}$$

9) Let V be the vector space $\mathbb{R}^3$ with basis $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$. Given the linear transformation below, find a formula for $T\left(\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}_B \right)$.

$$T: V \to \mathbb{R}^3_S$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$$

**Note that the actual definition of B didn't affect our answer, because T was encoded on V itself.

10) Let V be the vector space $\mathbb{R}^3$ with basis $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$.

Given the coordinate vector $[\vec{v}]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_B$, find $[\vec{v}]_S$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}$$

11) Let A be an 3×3 matrix. You know that it has eigenvalues $\lambda_1 = 2, \lambda_2 = 5, \lambda_3 = 10$. What else can you say?

(Maximum +10/-10 points) (You may list as many statements as you like, each insightful statement is worth +2 or -2 points.)

This is the big theorem. Here is a selection of statements that we can say. The key piece of information to notice is that $\lambda = 0$ is not an eigenvalue, as a 3×3 matrix can have only 3 eigenvalues. That means that A is nonsingular. Other things we can say are:

Statements on solutions and free variables:

The equation $A\vec{x} = \vec{b}$ has a free variable – FALSE

The equation $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b}$ - TRUE

The equation $A\vec{x} = \vec{0}$ has infinitely many solutions – FALSE

The equation $A\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as a solution - TRUE

Statements on linear dependence and independence of rows and columns:

The rows of A are linearly dependent – FALSE

The rows of A are linearly independent – TRUE

The columns of A are linearly independent – TRUE

The columns of A are linearly dependent – FALSE

Statements on the matrix itself

In echelon form, every row has a pivot – TRUE

In echelon form every column has a pivot – TRUE

In echelon form A has a row of zeroes – FALSE

The row space spans $\mathbb{R}^3$ – TRUE

The column space spans $\mathbb{R}^3$ – TRUE

The null space is nontrivial – FALSE

The matrix is invertible – TRUE

The matrix is a product of elementary matrices – TRUE

The matrix is singular – FALSE

The rank of the matrix is 3 – TRUE

The determinant of the matrix is zero – FALSE

Statements on the corresponding linear transformation

The corresponding linear transformation is one-to-one – TRUE

The corresponding linear transformation is onto – TRUE

Statements on eigenvalues and diagonalization

$\lambda = 0$ is not an eigenvalue - TRUE

The matrix is diagonalizable – TRUE