Name ____ Test 3, Fall 2020

Before you start, note that the last page contains some formulas that might help you on some problems. You can tear that page off for reference as you complete the test if you like.

1) Find the eigenvalues and eigenspaces of the matrix below. Please label or circle your answers. (15 points)

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

First we find the eigenvalues (Expand on the 2nd row):

$$\begin{vmatrix} x-1 & 0 & 2 \\ 0 & x+1 & 0 \\ 2 & 0 & x+2 \end{vmatrix} = (x+1)\begin{vmatrix} x-1 & 2 \\ 2 & x+2 \end{vmatrix} =$$

$$(x+1)\big((x-1)(x+2)-4\big) = (x+1)(x^2-x+2x-2-4)$$

$$= (x+1)(x^2+x-6) = (x+1)(x+3)(x-2)$$

Eigenvalues: $-1, 2, -3$.

Now for each eigenvalue we find its eigenspace:

$\lambda_1 = -1$:

$$\begin{bmatrix} -1-1 & 0 & 2 \\ 0 & -1+1 & 0 \\ 2 & 0 & -1+2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenspace: $span\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

$\lambda_1 = 2$:

$$\begin{bmatrix} 2-1 & 0 & 2 \\ 0 & 2+1 & 0 \\ 2 & 0 & 2+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenspace: $span\left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right)$

$\lambda_1 = -3$:

$$\begin{bmatrix} -3-1 & 0 & 2 \\ 0 & -3+1 & 0 \\ 2 & 0 & -3+2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -1 \end{bmatrix} \sim_R \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenspace: $span\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right)$

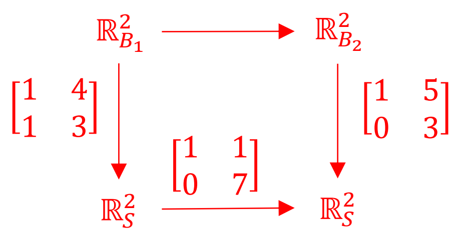
2) Let $B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$ and $B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$. Define the linear transformation $T: \mathbb{R}^2_S \to \mathbb{R}^2_S$ as below.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_S \right) = \begin{bmatrix} x_1 + x_2 \\ 7x_2 \end{bmatrix}_S$$

Find a formula for $[T]_{B_1}^{B_2}$, the matrix representation of T in the given bases. (15 points)

According to the diagram below, we get:

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$$



3) Answer the questions below. (3 points each)

(A) Let A be a 6×6 matrix with eigenvalues $0, 2, 2, 4, 5$, and 6. How many solutions does $A\vec{x} = \vec{0}$ have?

∞ because A has $\det(A) = 0$, which means that $A\vec{x} = \vec{0}$ has a free variable.

(B) Let A be a 4×6 matrix whose corresponding linear transformation is onto. What is the rank of A?

4 – because being onto means that $T(\vec{x}) = A\vec{x} = \vec{b}$ can be solved for any choice of $\vec{b}$. So when row reduced, $\vec{A}$ will have a pivot in each of its 4 rows.

(C) Let A be a 5×4 matrix. And let $\vec{b}_1$ and $\vec{b}_2$ both be nonzero vectors such that $A\vec{x} = \vec{b}_1$ has a unique solution, but $A\vec{x} = \vec{b}_2$ has no solutions. What can be said about the dimension of the row space of A?

4 because:
Having a unique solution means that the system has no free variables, so the rank of A is at least the number of columns, 4.
Having no solution means that the row reduced matrix has a row of zeroes, so the rank of A is less than 5.
Hence the rank of A is 4, so the dimension of the row space is 4.

(D) Let A be a 5×4 matrix. And let $\vec{b}_1$ and $\vec{b}_2$ both be nonzero vectors such that $A\vec{x} = \vec{b}_1$ has a multiple solutions, but $A\vec{x} = \vec{b}_2$ has no solutions. What can be said about the dimension of the row space of A?

Between 1 and 3 because:
Having multiple solution means that the system has free variables, so the rank of A is less than the number of columns, 4.
Having no solution means that the row reduced matrix has a row of zeroes, so the rank of A is less than 5.
The rank must be at least 1 because the first nonhomogeneous system has solutions. Hence the dimension of the row space is 1, 2, or 3 based on what we know.

(E) If $T: \mathbb{R}^5 \to \mathbb{R}^{17}$ is a linear transformation that is one-to-one and A is the matrix that represents T, what is the dimension of the null space of A?

0, because being one-to-one tells us that A has no free variables, so it has rank 5, the number of columns. There are no columns left to contribute to the null space dimension.

4) Find a formula for $\begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & -2 \end{bmatrix}^{10}$ that requires at most 3 matrix multiplications. It may involve other arithmetic or matrix operations. (10 points)

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} (-3)^{10} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1024 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}^{-1}$$

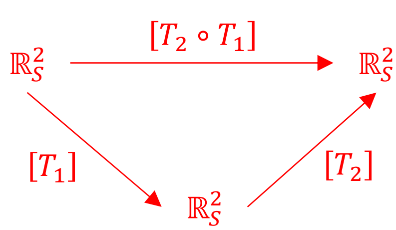
5) Let T_1 and T_2 be defined as below. Find a formula for $[T_2 \circ T_1]$. (5 points)

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $[T_1] = \begin{bmatrix} 1 & 8 \\ 2 & 7 \end{bmatrix}$.

$T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $[T_2] = \begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix}$.

We don't really need the diagram for this, but I'll draw it just to illustrate it.

$$[T_2 \circ T_1] = \begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 2 & 7 \end{bmatrix}$$



6) Let T be defined as below. Find the kernel of T. (10 points)

$T_1: \mathbb{R}^4 \to \mathbb{R}^3$ is given by $[T] = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 4 & 8 & 4 \\ 1 & 4 & 8 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 4 & 8 & 4 \\ 1 & 4 & 8 & 4 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\ker(T) = span\left(\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

7) Multiply the two matrices below. (10 points)

$$\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2+2 & 5-2 \\ 8-3 & 20+3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 23 \end{bmatrix}$$

8) Row reduce the matrix below. (5 points)

$$\begin{bmatrix} 2 & 8 & 4 \\ 4 & 10 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 8 & 4 \\ 4 & 10 & 8 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 4 & 2 \\ 4 & 10 & 8 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 4 & 2 \\ 0 & -6 & 0 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 0 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

9) Answer yes (Y) or no (N) to the following questions. Is it possible to multiply these matrices?.
(2 points each)

(Y) N (A) $\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 2 & 8 & 4 \\ 4 & 10 & 8 \end{bmatrix}$

Y (N) (B) $\begin{bmatrix} 2 & 8 & 4 \\ 4 & 10 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$

(Y) N (C) The matrix on the left is 5×6; the matrix on the right is 6×7

10) Is $\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$? If so, is it unique? (9 points)

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 6 & 4 \\ 5 & 7 & 9 & 7 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Yes it is a linear combination, no it is not unique.

Yes because the column for $\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$ does not have a pivot.

No because a column before it corresponds to a free variable.

The following formulas might or might not be useful on some problems on this test.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & -2 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 7 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 8 \\ 2 & 7 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0.8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 6 & 4 \\ 5 & 7 & 9 & 7 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 4 & 8 & 4 \\ 1 & 4 & 8 & 4 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$