Name ____ Test 1, Fall 2021

1) Multiply the two matrices below or state why they cannot be multiplied. (15 points)

$$\begin{bmatrix} 1 & 6 \\ 3 & 2 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 2 & -1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 14 & -3 & 33 \\ 10 & 7 & 19 \\ 18 & 19 & 31 \end{bmatrix}$$

2) Find the null space of the matrix below. (15 points)

$$\begin{bmatrix} 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 6 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$x_2 - 2x_4 = 0$$
$$x_3 + 6x_4 - 4x_6 = 0$$
$$x_5 + x_6 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_4 \\ -6x_4 + 4x_6 \\ x_4 \\ -x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 2 \\ -6 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ -1 \\ 1 \end{bmatrix} x_6$$

The null space is then:

$$span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

Half credit: Minor mistake but something of merit such as missing one variable, improper notation

No credit: major mistake such as improper equations or vectors in the wrong dimension

3) Reduce the matrix below to reduced row echelon form. (15 points)

$$\begin{bmatrix} 0 & 0 & 3 & 2 & 1 \\ 1 & 2 & 9 & 6 & 3 \\ 2 & 4 & 6 & 4 & 2 \\ 0 & 0 & 12 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 3 & 2 & 1 \\ 1 & 2 & 9 & 6 & 3 \\ 2 & 4 & 6 & 4 & 2 \\ 0 & 0 & 12 & 4 & 4 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 2 & 9 & 6 & 3 \\ 0 & 0 & 3 & 2 & 1 \\ 2 & 4 & 6 & 4 & 2 \\ 0 & 0 & 12 & 4 & 4 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 2 & 9 & 6 & 3 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & -12 & -8 & -4 \\ 0 & 0 & 12 & 4 & 4 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 2 & 9 & 6 & 3 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 4 & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ $\quad R_3 \to R_3 - 2R_1$ $\quad R_3 \to R_3 + 4R_2$

$$\sim_R \begin{bmatrix} 1 & 2 & 9 & 6 & 3 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \to R_4 - 4R_2$ $\quad R_1 \to R_1 - 3R_2$ $\quad R_2 \to \frac{1}{3}R_2$ $\quad R_3 \leftrightarrow R_4$

$R_3 \to -\frac{1}{4}R_3$

$$\sim_R \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - \frac{2}{3}R_3$

4) Answer the questions below (3 points each)

(A) Let A be a 3×3 matrix such that $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ has a unique solution.

Is A a product of elementary matrices?

Yes

(B) Let A be a 5×7 matrix. When row reduced, it has 4 pivots. What is the dimension of the column space?

4

(C) Let A be a 6×4 matrix. When row reduced, it has 3 pivots.

How many solutions can $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ have?

0 or ∞

(D) Let A be a 4×3 matrix such that $A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has a unique solution, but $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ has no solutions. What is the rank of A?

3

(E) Let A be a 12×7 matrix whose row space has dimension 4. When row reduced, how many pivots does it have?

4

5) Find the inverse of the matrix below. (10 points)

$$\begin{bmatrix} 9 & 18 & 27 \\ 2 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccccc} 9 & 18 & 27 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim_R \left[\begin{array}{cccccc} 1 & 2 & 3 & \frac{1}{9} & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim_R \left[\begin{array}{cccccc} 1 & 2 & 3 & \frac{1}{9} & 0 & 0 \\ 0 & -1 & -1 & -\frac{2}{9} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to \frac{1}{9} R_1$ $\quad$ $R_2 \to R_2 - 2R_1$

$$\sim_R \left[\begin{array}{cccccc} 1 & 0 & 1 & -\frac{3}{9} & 2 & 0 \\ 0 & -1 & -1 & -\frac{2}{9} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim_R \left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{3}{9} & 2 & -1 \\ 0 & -1 & 0 & -\frac{2}{9} & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim_R \left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{3}{9} & 2 & -1 \\ 0 & 1 & 0 & \frac{2}{9} & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 + 2R_2$ $\quad$ $R_2 \to R_2 + R_3$ $\quad$ $R_2 \to -R_2$

$$\begin{bmatrix} 9 & 18 & 27 \\ 2 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{9} & 2 & -1 \\ \frac{2}{9} & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The following row reduction may or may not be useful for the problems on this page.

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 3 & 3 & 1 \\ 2 & 1 & 5 & 1 \\ 1 & -5 & -3 & 2 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6) Determine whether or not the collection below is a vector space. Why? (5 points)

$$\left\{ \begin{bmatrix} x_1 + 2x_2 + 4x_3 + x_4 \\ 3x_2 + 3x_3 + x_4 \\ 2x_1 + x_2 + 5x_3 + x_4 \\ x_1 - 5x_2 - 3x_3 + 2x_4 \end{bmatrix} : x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$$

It is, because it is the span of three vectors, which is by construction a vector space.

7) Can $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ can be written as a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 1 \\ -5 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 3 \\ 5 \\ -3 \end{bmatrix}$. Why? (5 points)

No. Look at the row reduced version, the pivot in the 4th column tells us that it is linearly independent from the first three vectors.

8) Find the row space of the matrix below. Do not include redundant vectors. (5 points)

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 3 & 3 & 1 \\ 2 & 1 & 5 & 1 \\ 1 & -5 & -3 & 2 \end{bmatrix}$$

$$span\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T \right)$$

9) Use the information below to solve the system of equations below. You do not need to simplify your answer. (5 points)

$$\begin{bmatrix} 12 & 7 & 3 \\ 20 & 13 & 6 \\ 3 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -12 \\ 1 & -3 & 16 \end{bmatrix}$$

$$\begin{aligned} 12x + 7y + 3z &= 2 \\ 20x + 13y + 6z &= 1 \\ 3x + 2y + z &= 4 \end{aligned}$$

$$\begin{bmatrix} 12 & 7 & 3 \\ 20 & 13 & 6 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -12 \\ 1 & -3 & 16 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

10) How many solutions does the equation below have? (5 points)

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

∞

11) Find the product below. (5 points)

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 2 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & -1\\0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}3 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2 & 2 & 1\\3 & 4 & 4 & 4 & 3\\5 & 6 & 6 & 6 & 5\\1 & 2 & 3 & 4 & 5\\1 & 1 & 2 & 2 & 3\end{bmatrix}$$

$$\begin{bmatrix}3 & 6 & 6 & 6 & 3\\6 & 8 & 8 & 8 & 6\\5 & 6 & 6 & 6 & 5\\0 & 1 & 1 & 2 & 2\\1 & 1 & 2 & 2 & 3\end{bmatrix}$$

(Use elementary matrices!)