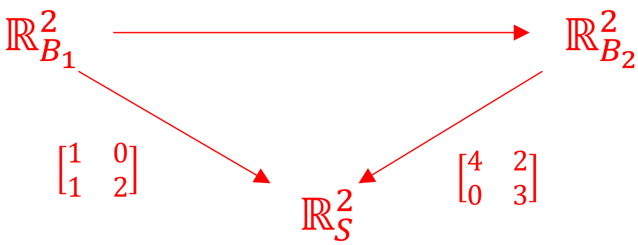
Name ____ Test 2, Fall 2021

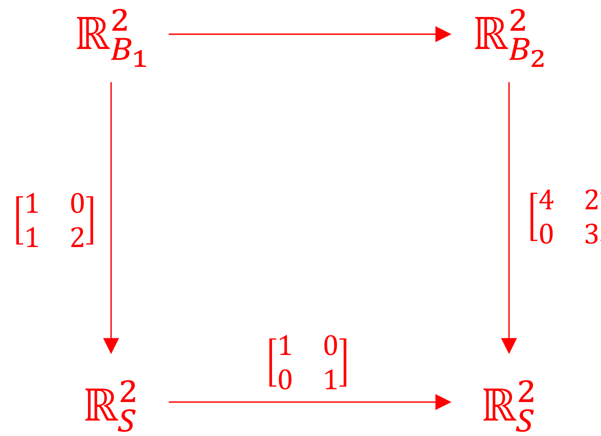
1) Given the basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\vec{x}_B = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}_B$, find $[\vec{x}]_S$. (10 points)

(Not just a formula; actually find it)

$$2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \vec{0} + 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix}$$

2) Let $B_1 = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right\}$ and $B_2 = \left\{\begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$. Draw an appropriate diagram representing this information that relates it to the standard basis. (10 points)





Either of these are acceptable. I prefer the one on the right because it generalizes to linear transformations better. In either case, we get the same answer for $[I]_{B_1}^{B_2}$

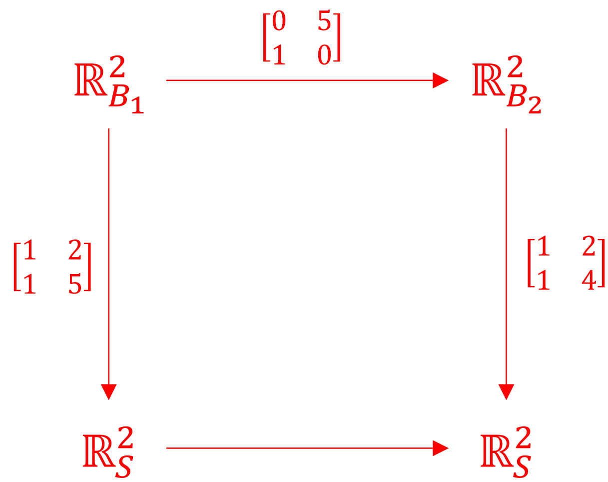
3) Find $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 3 & 5 & 0 \\ 2 & 0 & 0 & 7 \end{vmatrix}$. (15 points)

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 3 & 5 & 0 \\ 2 & 0 & 0 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 4 \\ 3 & 5 & 0 \\ 0 & 0 & 7 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 & 4 \\ 0 & 5 & 0 \\ 2 & 0 & 7 \end{vmatrix} = 2\begin{vmatrix} 5 & 7 \\ 0 & 7 \end{vmatrix} + 4\begin{vmatrix} 3 & 5 \\ 0 & 0 \end{vmatrix} - 2 \cdot 4 \cdot \begin{vmatrix} 0 & 5 \\ 2 & 0 \end{vmatrix}$$

$$= 2(35) + 0 - 8 \cdot (-10) = 70 + 80 = 150$$

4) Let $B_1 = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}\right\}$ and $B_2 = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$. Define the linear transformation $T: \mathbb{R}^2_{B_1} \to \mathbb{R}^2_{B_2}$ via the equation below. Find a formula for $[T]^S_{B_1}$. (10 points)

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}_{B_1}\right) = \begin{bmatrix}5x_2\\x_1\end{bmatrix}_{B_2}$$



$[T]^S_{B_1} = \begin{bmatrix}1 & 2\\1 & 4\end{bmatrix}\begin{bmatrix}0 & 5\\1 & 0\end{bmatrix}$

5) Answer the questions below (3 points each)

(A) Let A be a 3×3 matrix such that $A\vec{x} = \vec{0}$ has one free variable. What is $|A|$?

0

(B) Let A be a 3×5 matrix such that, when row reduced, has only 1 pivot. What is the dimension of the null space of A?

4

(C) Let A be a 5×3 matrix and T be the corresponding linear transformation. Assume T is one-to-one. How many pivots does A have, when row reduced?

3

(D) Let $A\vec{x} = \vec{0}$ be a system of equations that has multiple solutions. Is the corresponding system of linear transformation one-to-one?

No.

(E) Let A be a 11×7 matrix. There are 6 linearly independent rows. What is the rank of A?

6

6) Row reduce the matrix $\begin{bmatrix} 1 & 3 & 2 & 3 \\ 0 & 0 & 3 & 0 \\ 2 & 6 & 4 & 5 \end{bmatrix}$ to reduced echelon form. (10 points)

$$\begin{bmatrix} 1 & 3 & 2 & 3 \\ 0 & 0 & 3 & 0 \\ 2 & 6 & 4 & 5 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 3 & 2 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 3 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\sim_R \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7) Find the determinant of $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$. (5 points)

$$14 - 15 = -1$$

8) Is the collection of vectors below a basis for some vector space? (5 points)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

No, it is linearly dependent.

9) Find the kernel of the linear transformation given by the linear transformation below. (5 points)

$$T: \mathbb{R}^4 \to \mathbb{R}^3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - 3x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

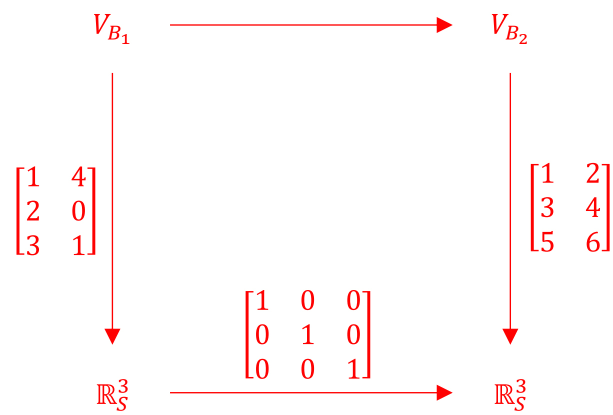
$$[T] = \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\ker(T) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

10) Given $T: \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, what is a formula for $[T^{-1}]$? (5 points)

$$[T^{-1}] = \left(\begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 2 \end{bmatrix}$$

11) Let $B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $B_2 = \left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$. Find a formula for the change of basis matrix $[I]_{B_1}^{B_2}$ that changes basis B_1 into basis B_2. (5 points)



$$[I]_{B_1}^{B_2} = \left(\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$$

12) Use Cramer's Rule to find a formula for the solution to x_3. (5 points)

$$\begin{bmatrix} 1 & 2 & 4 \\ 6 & 7 & 2 \\ 0 & 9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 12 \end{bmatrix}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & 2 \\ 6 & 7 & 3 \\ 0 & 9 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 6 & 7 & 2 \\ 0 & 9 & 0 \end{vmatrix}}$$