Name ____ Test 3, Fall 2021

1) Given the matrix below, find its eigenspaces. Circle or box your answer(s). (15 points)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$xI - A = \begin{bmatrix} x-1 & -1 & 0 \\ 0 & x-2 & 0 \\ 0 & -1 & x-2 \end{bmatrix}$$

$$|xI - A| = \begin{vmatrix} x-1 & -1 & 0 \\ 0 & x-2 & 0 \\ 0 & -1 & x-2 \end{vmatrix} = (x-2)\begin{vmatrix} x-1 & 0 \\ 0 & x-2 \end{vmatrix} = (x-2)(x-1)(x-2)$$

Eigenvalues: 1, 2, 2.

$\lambda_1 = 1$

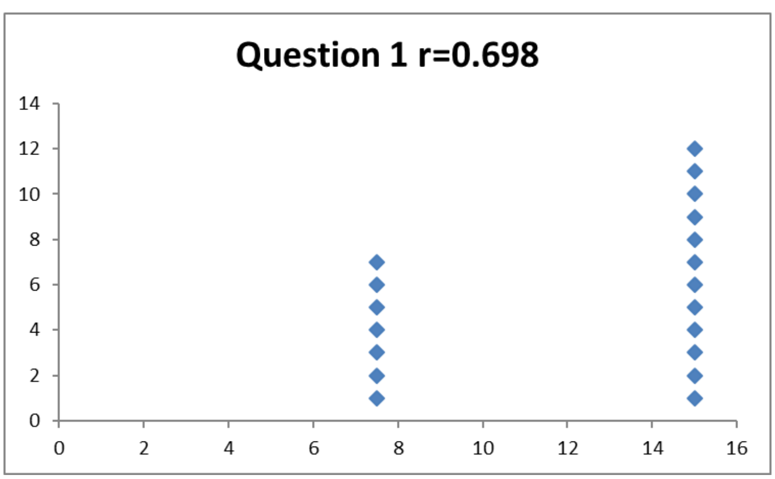
$$[1I - A] = \begin{bmatrix} 1-1 & -1 & 0 \\ 0 & 1-2 & 0 \\ 0 & -1 & 1-2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \sim_R \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenspace: $span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

$\lambda_1 = 2$

$$[2I - A] = \begin{bmatrix} 2-1 & -1 & 0 \\ 0 & 2-2 & 0 \\ 0 & -1 & 2-2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenspace: $span\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$



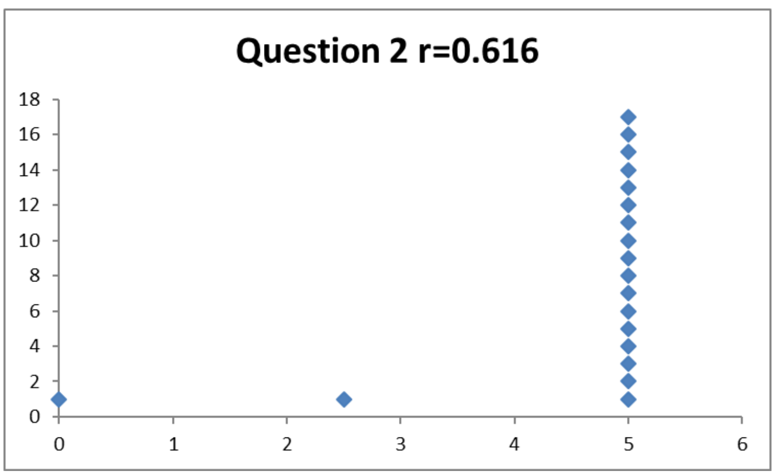
2) Find the eigenvalues of the linear transformation below. (5 points)

$$T: \mathbb{R}^4 \to \mathbb{R}^4$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} 2x_1 \\ 3x_2 \\ 5x_3 \\ x_4 \end{bmatrix}$$

$$A = [T] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|xI - A| = \begin{vmatrix} x-2 & 0 & 0 & 0 \\ 0 & x-3 & 0 & 0 \\ 0 & 0 & x-5 & 0 \\ 0 & 0 & 0 & x-1 \end{vmatrix} = (x-2)(x-3)(x-5)(x-1)$$

Eigenvalues: $1, 2, 3, 5$.



3) Given the basis below, find an orthogonal version of the basis. (10 points)

$$\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\5\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\\-13\end{bmatrix}\right\}$$

$$\vec{b}_1=\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$$

$$proj_{\vec{b}_1}\left(\begin{bmatrix}0\\1\\5\\1\end{bmatrix}\right)=\frac{6}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix}=\begin{bmatrix}2\\2\\2\\0\end{bmatrix}$$

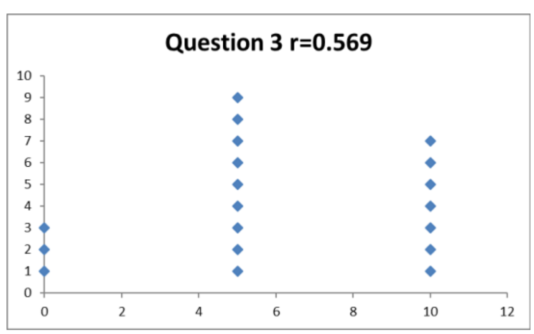
$$\vec{b}_2=\begin{bmatrix}0\\1\\5\\1\end{bmatrix}-\begin{bmatrix}2\\2\\2\\0\end{bmatrix}=\begin{bmatrix}-2\\-1\\3\\1\end{bmatrix}$$

$$proj_{\vec{b}_1}\left(\begin{bmatrix}1\\0\\0\\-13\end{bmatrix}\right)=\frac{1}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix}=\begin{bmatrix}1/3\\1/3\\1/3\\0\end{bmatrix}$$

$$proj_{\vec{b}_2}\left(\begin{bmatrix}1\\0\\0\\-13\end{bmatrix}\right)=\frac{-15}{15}\begin{bmatrix}-2\\-1\\3\\1\end{bmatrix}=\begin{bmatrix}2\\1\\-3\\-1\end{bmatrix}$$

$$\vec{b}_3=\begin{bmatrix}1\\0\\0\\-13\end{bmatrix}-\begin{bmatrix}1/3\\1/3\\1/3\\0\end{bmatrix}-\begin{bmatrix}2\\1\\-3\\-1\end{bmatrix}=\begin{bmatrix}-4/3\\-4/3\\8/3\\-12\end{bmatrix}$$

$$B=\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix},\begin{bmatrix}-2\\-1\\3\\1\end{bmatrix},\begin{bmatrix}-4/3\\-4/3\\8/3\\-12\end{bmatrix}\right\}$$



4) Answer the questions below (3 points each)

(A) Let A be a 3×3 matrix with eigenvalues $0, 1$, and 4. What is $|A|$?

0

(B) Let A be a 3×3 matrix with eigenvalues $0, 1$, and 1 again. What is the dimension of the null space of A?

1

(C) Let $T: \mathbb{R}^6 \to \mathbb{R}^{10}$ be a one-to-one linear transformation. What is the dimension of $\ker(T)$.

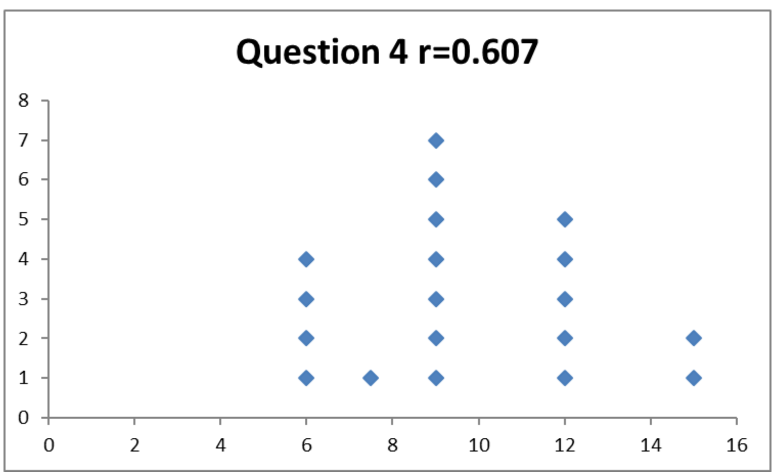
1

(D) Let A be a 12×4 matrix whose columns are linearly independent. Is the corresponding linear transformation onto?

No.

(E) Let $T: \mathbb{R}^5 \to \mathbb{R}^5$ be a linear transformation that is onto. Is it one-to-one?

Yes

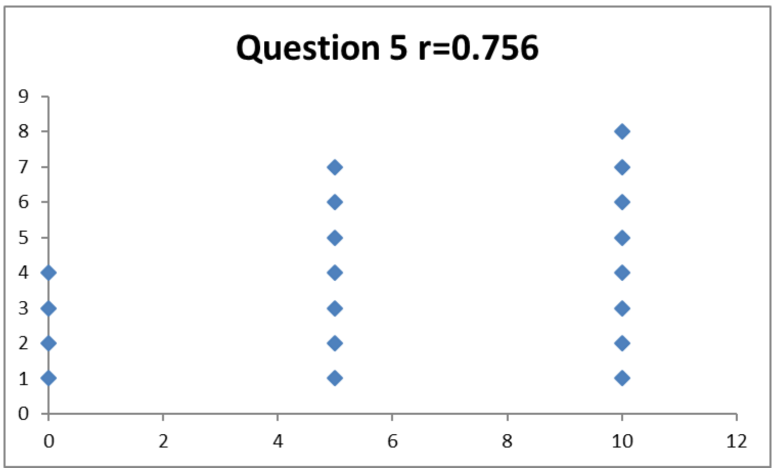


5) Find the null space of the matrix below. (10 points)

$$\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$2x_1 + 3x_4 = 0$
$x_2 = 0$
$x_4 = 0$

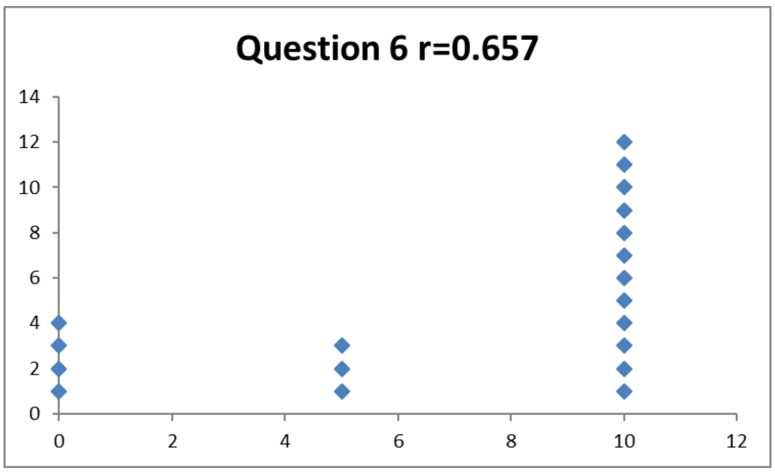
$$span\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$$



6) Consider the basis below and the coordinate vector $\begin{bmatrix}2\\1\\3\end{bmatrix}_B$. Find a formula for this same vector, represented in the standard coordinates. (10 points)

$$B = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\5\end{bmatrix}\right\}$$

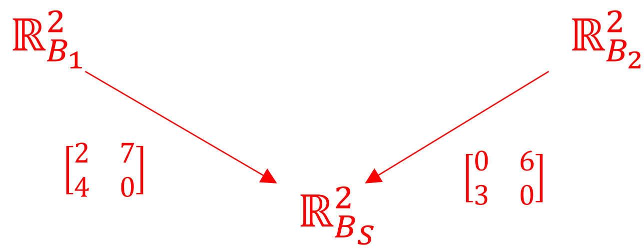
$$\begin{bmatrix}1 & 3 & 0\\1 & 2 & 0\\0 & 0 & 5\end{bmatrix}\begin{bmatrix}2\\1\\3\end{bmatrix}$$

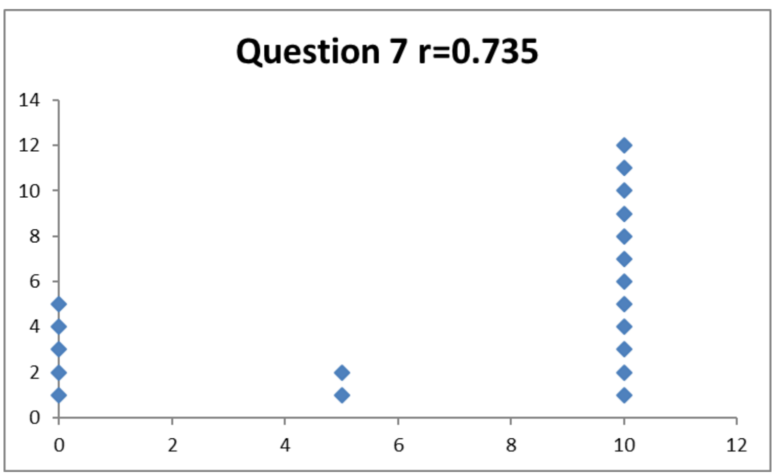


7) Consider the two bases below and the vector $[\vec{x}]_{B_2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{B_2}$. Find a formula for $[\vec{x}]_{B_1}$. (10 points)

$$B_1 = \left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix} \right\} \quad B_2 = \left\{ \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 2 & 7 \\ 4 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 6 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$





8) Find the diagonalization of the matrix below. (5 points)

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$xI - A = \begin{bmatrix} x-1 & -2 \\ 0 & x-2 \end{bmatrix}$

$|xI - A| = \begin{bmatrix} x-1 & -2 \\ 0 & x-2 \end{bmatrix} = (x-1)(x-2)$

$\lambda = 1$:

$\begin{bmatrix} 1-1 & -2 \\ 0 & 1-2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix} \sim_R \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Eigenvector: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

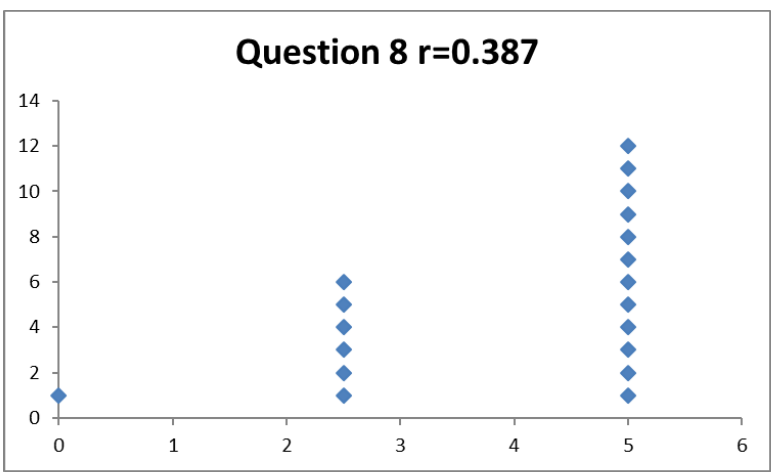
$\lambda = 2$:

$\begin{bmatrix} 2-1 & -2 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

Eigenvector: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

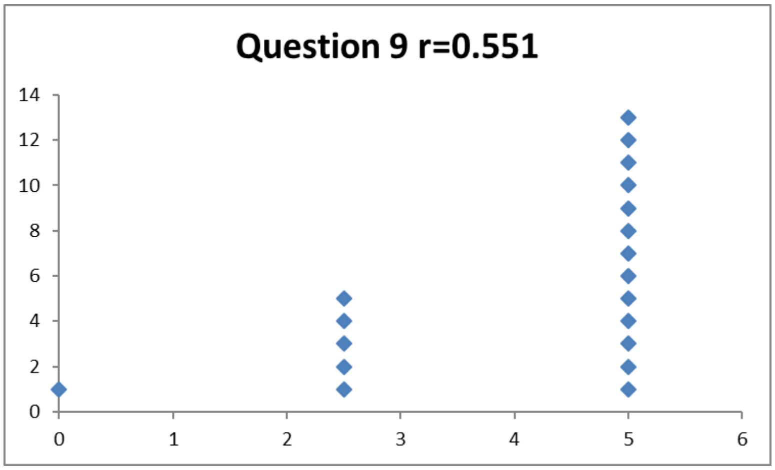
Diagonalization:

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$



9) Find a formula for $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}^{1000}$ that involves no more than 5 matrix multiplications. (5 points)

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}^{1000} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{1000} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$



10) If it is known that $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}6\\3\end{bmatrix}$, find the rule for T. (5 points)

$[T] = \begin{bmatrix}2 & 6\\4 & 3\end{bmatrix}$

There are various ways to correctly express the rule. The rule tells us how to get from the input vector to the output vector. They all involve an equals sign (=) or the mapsto (↦) sign. For full credit you must have one of these.

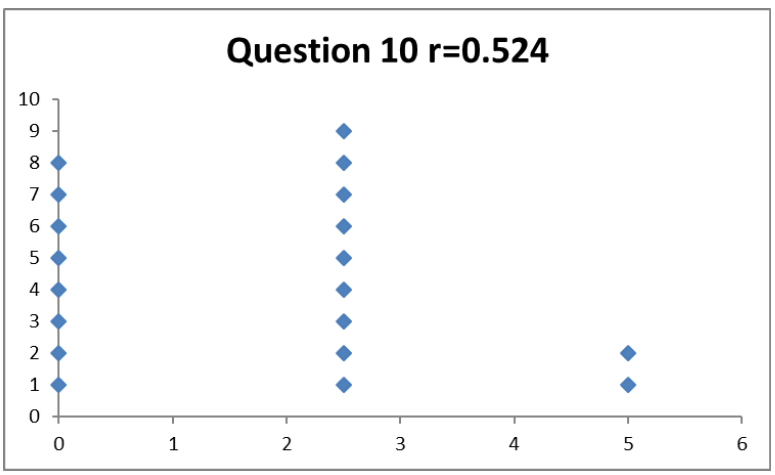
$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}2 & 6\\4 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}2x_1 + 6x_2\\4x_1 + 3x_2\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto \begin{bmatrix}2 & 6\\4 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Or

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto \begin{bmatrix}2x_1 + 6x_2\\4x_1 + 3x_2\end{bmatrix}$$

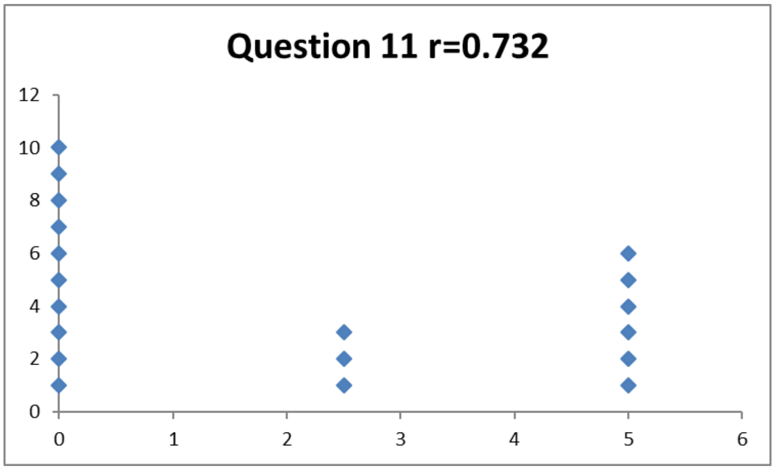


11) Find a formula for the angle between the vectors $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 5 \end{bmatrix}$. (5 points)

Recall the formula:

$$\vec{u} \bullet \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta)$$

$$\theta = \cos^{-1}\left(\frac{12 + 20}{\sqrt{20}\sqrt{36 + 20}}\right)$$



12) Write down span$\left(\left\{\begin{bmatrix}2\\4\end{bmatrix},\begin{bmatrix}6\\5\end{bmatrix}\right\}\right)$ using set builder notation. (5 points)

$\left\{\begin{bmatrix}2\\4\end{bmatrix}x_1+\begin{bmatrix}6\\5\end{bmatrix}x_2 : x_1, x_2 \in \mathbb{R}\right\}$