Name __Solutions____ Discrete II, Test 1

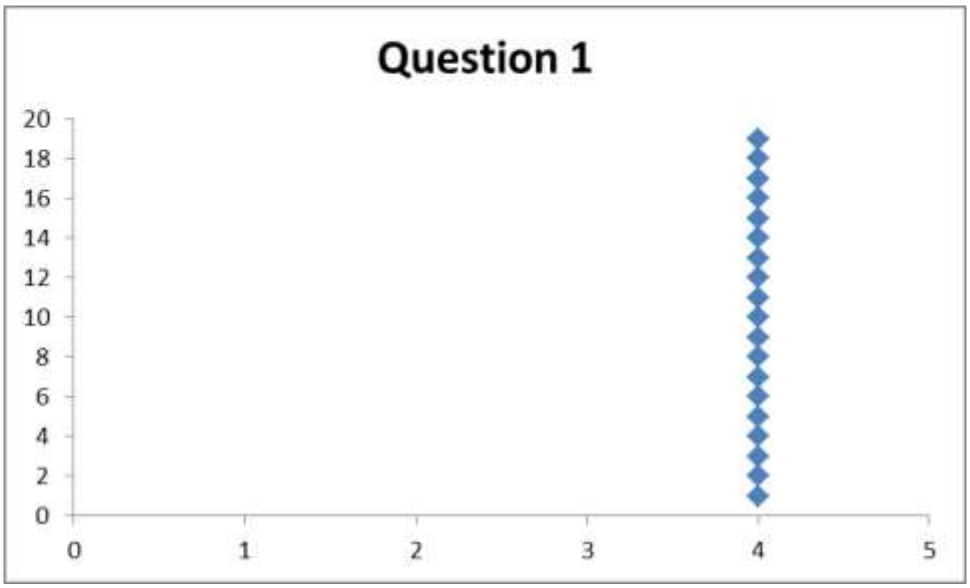
Please do not simplify any answers; no calculators are allowed.

1) A man has 5 shirts, 3 pants, and 23 shoes. How many outfits are possible? (4 points)

There are three things and we must choose something from each, hence this is a multiplication problem:

$$5 \cdot 3 \cdot 23$$

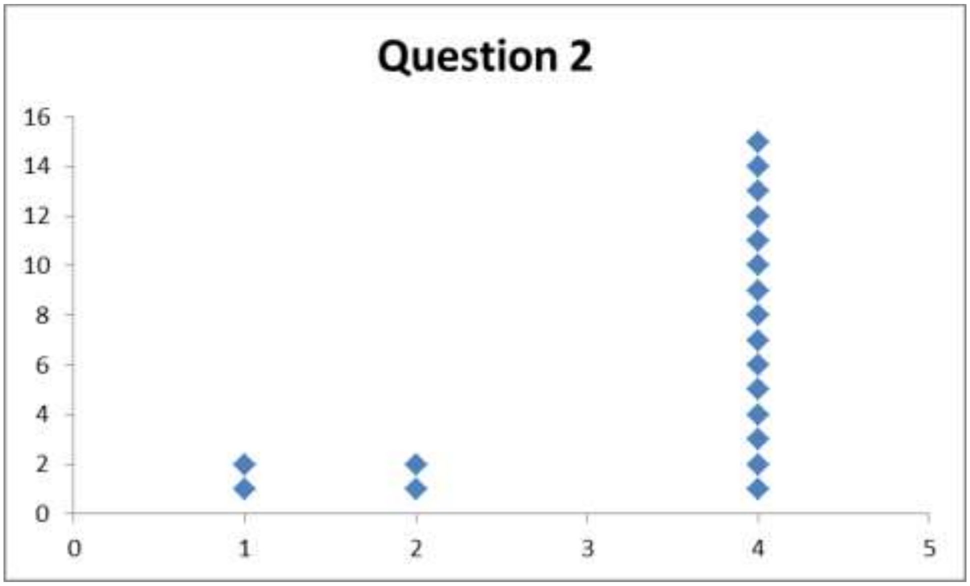
Clarified during test: 23 pairs of shoes.



2) How many 12 bit strings have exactly one 1? (4 points)

A bit string consists of 0's and 1's. If there is exactly one 1, then there are 11 0's. We can choose where the 1 is, and everything else is a 0:

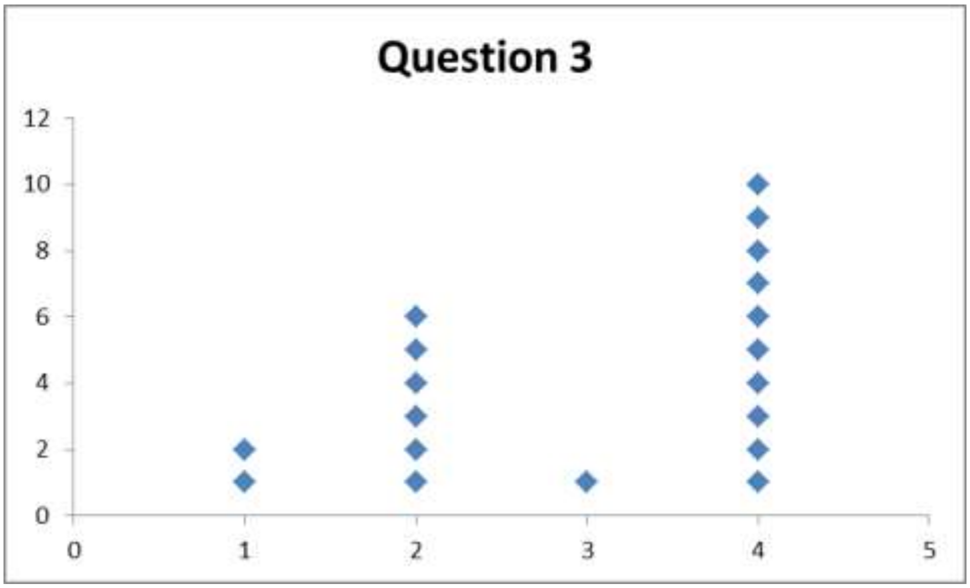
$$\binom{12}{1} = 12$$



3) At a renaissance fair, three people are needed for king, queen, and jester. There are 14 people that want to volunteer. How many ways can the king, queen, and jester be selected? (4 points)

There are 14 people to choose from, and we are selecting 3 of them with order because we can tell the difference between a king, queen, and jester. There is no replacement because a person cannot serve multiple rules, so we subtract one from each selection. We must choose all three, so it is a multiplication problem.

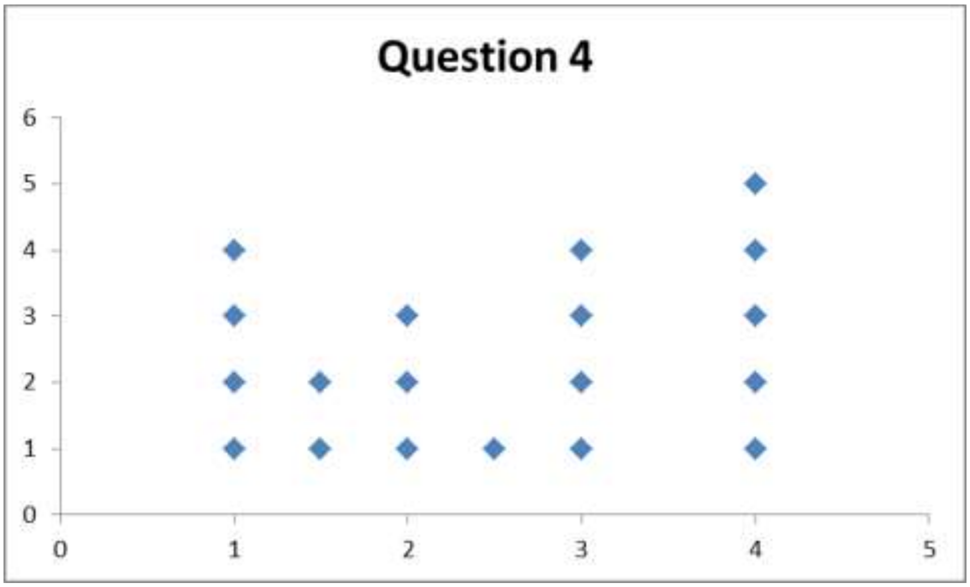
$$14 \cdot 13 \cdot 12$$



4) In bridge you are dealt 13 cards from a standard deck of 52 cards. How many hands are there that contain all 4 aces? (4 points)

There are 4 aces, we must choose all 4. Then there are 48 cards left, and we must choose 9 of them:

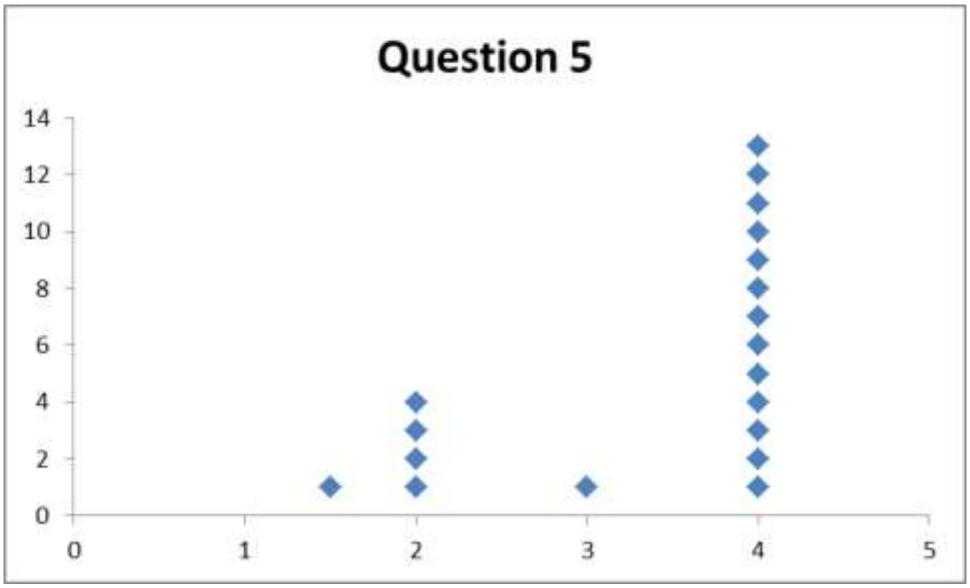
$$\binom{4}{4}\binom{48}{9} = 1 \cdot \frac{48!}{39!\,9!}$$



5) Find the number of solutions to $x_1 + x_2 + x_3 + x_4 = 12$, where each x_i is a nonnegative integer. (4 points)

This is a "stars and bars" problem: we must distribute 12 units into 4 boxes. But those 4 boxes are separated by 3 plus signs. Hence we have a total of 15 objects, and we must choose 12 of them to be the units. (Or 3 of them to be the plus signs):

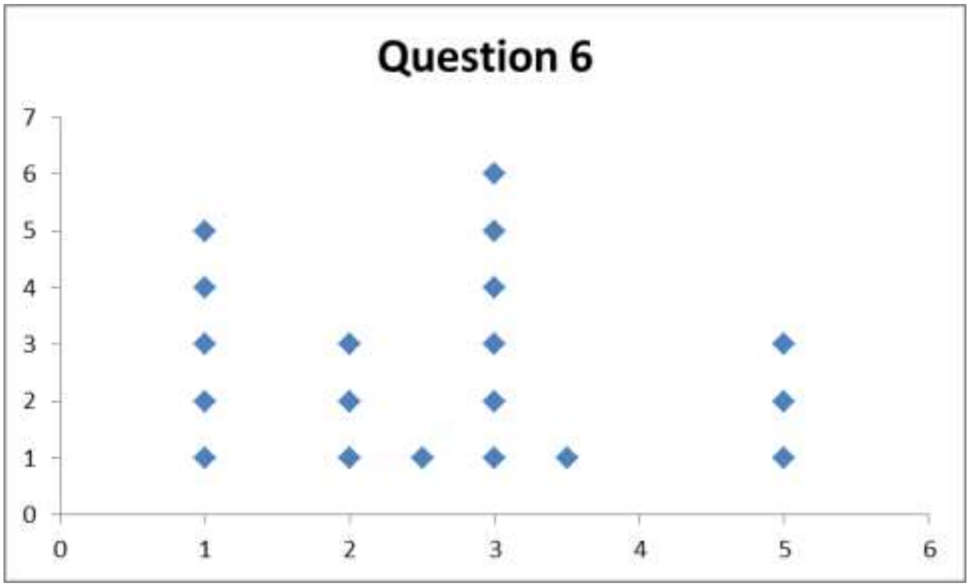
$$\binom{15}{3} = \binom{15}{12}$$



6) A fair coin is flipped 10 times. What is the probability that it shows exactly 1 heads? (5 points)

First we must select which one is the heads, then multiply by the probability of the heads, and the probability of all the tails:

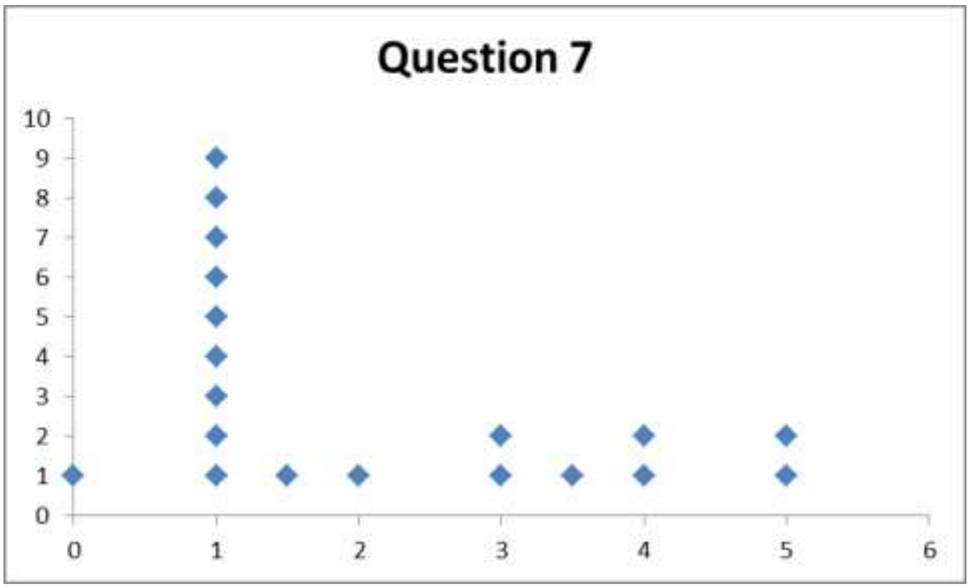
$$\binom{10}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^9 = \frac{10}{2^{10}}$$



7) An unfair coin is flipped 10 times. What is the probability that it shows exactly 1 heads? This coin has a $\frac{2}{3}$ probability of landing on heads. (5 points)

First we must select which one is the heads, then multiply by the probability of the heads, and the probability of all the tails:

$$\binom{10}{1}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^9 = \frac{20}{3^9}$$



8) Two dice are rolled. What is the probability of getting a sum of 6 or 8 given that at least one die shows 2? (5 points)

Knowing that one die shoes at least a 2, there are 4 favorable outcomes, and 9 possible outcomes, as shown below.

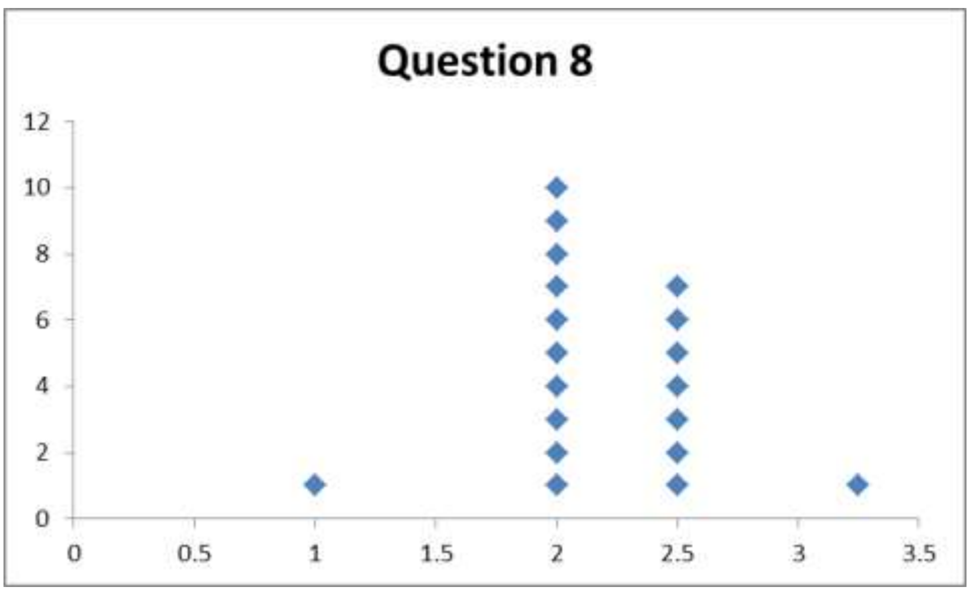
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Hence the probability is:

$$\frac{4}{9}$$

Or, if you use the formula:

$$\frac{P(\text{sum of 6 or 8} \cap \text{at least one 2})}{P(\text{at least one 2})} = \frac{4/36}{9/36}$$

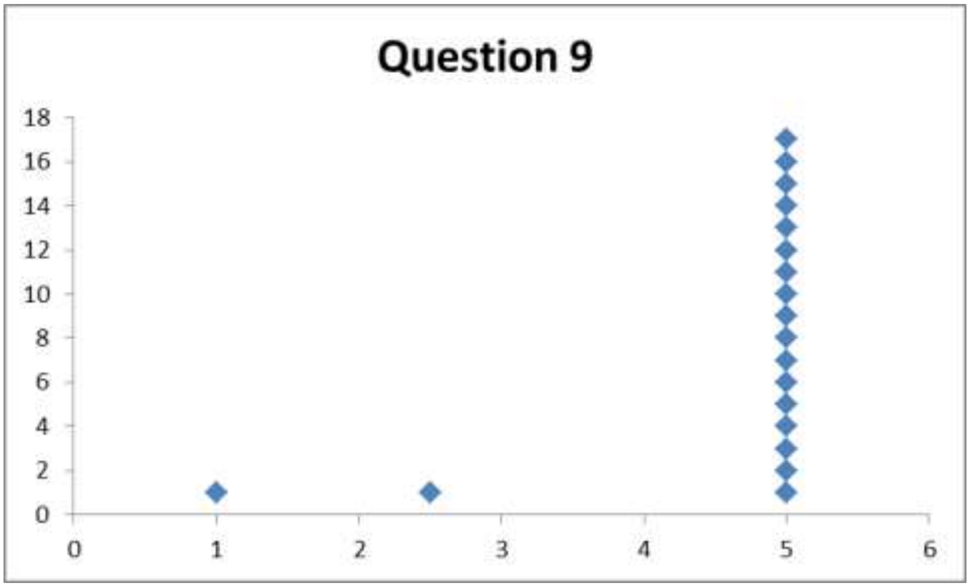


9) A fair coin is flipped 10,000 times. Is it possible that it comes up heads every time? (5 points)

Yes, although it is quite unlikely. In particular, the probability of such an event occurring is:

$$\frac{1}{2^{10000}} = 0.000 \cdots 000501237$$

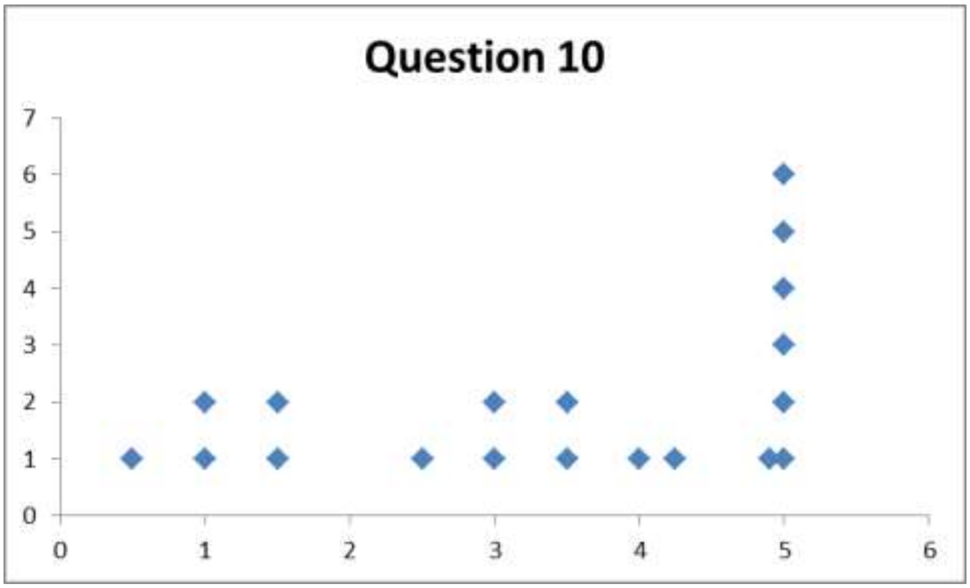
3004 zeroes omitted



10) $100 is invested in an account with 5% annually compounded interest. How much money is in the account after 30 years? After n years? (5 points)

This is a geometric progression with starting value 100 and rate 1.05.

After 30 years: $a_{30} = 100 \cdot (1.05)^{30}$
After n years: $a_n = 100 \cdot (1.05)^n$



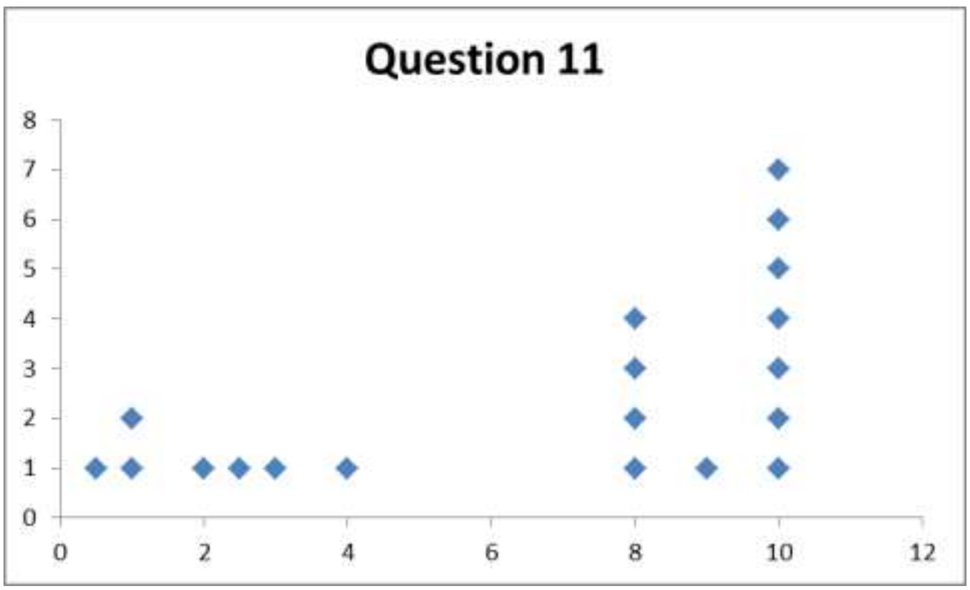
11) Consider the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$. Find a closed form expression for the general solution a_n. (10 points)

This is a second order linear recurrence relation. We will use the characteristic equation to find x_1, x_2. The solution is then $a_n = d_1 x_1^n + d_2 x_2^n$.

$$x^2 = 3x - 2$$
$$x^2 - 3x + 2 = 0$$
$$(x-2)(x-1) = 0$$
$$x_1 = 1, x_2 = 2$$

Hence the general solution is:

$$a_n = d_1 + d_2(2^n)$$



12) Building on the previous problem, if $a_0 = 0$ and $a_1 = 1$, what is a_n? (5 points)

We know that the general solution is $a_n = d_1 + d_2(2^n)$. We will now plug in $n = 0$ and $n = 1$, then solve this system for d_1 and d_2:

$$d_1 + d_2 = 0$$
$$d_1 + 2d_2 = 1$$

$$\therefore d_1 = -d_2$$

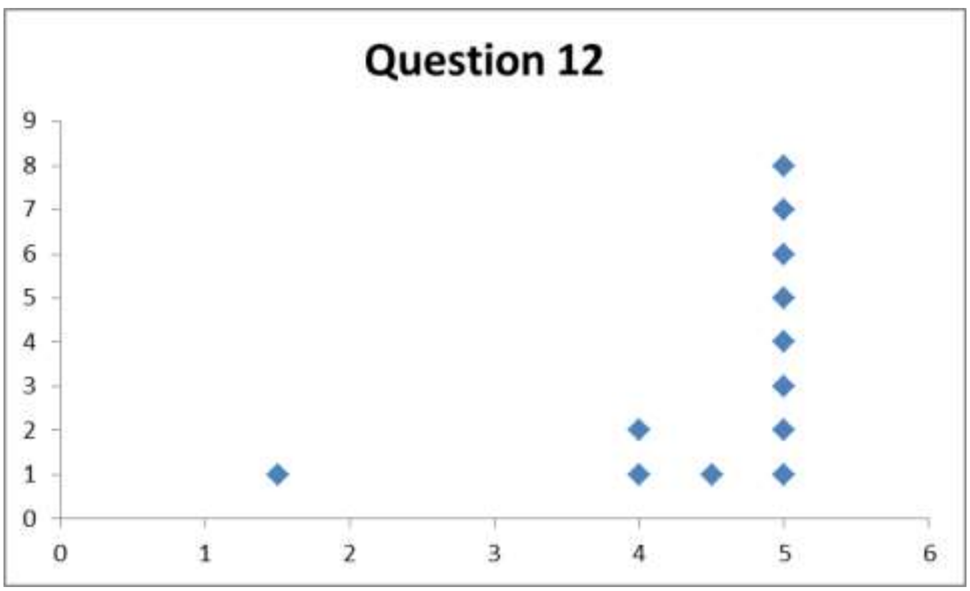
$$\therefore -d_2 + 2d_2 = 1$$
$$\therefore d_2 = 1$$

$$\therefore d_1 = -1$$

Thus, the particular solution is:

$$a_1 = -1 + 2^n$$

Note that this problem was only counted if the answer from (11) involved the variable n. Otherwise you didn't have a general solution and can't hope to find a particular solution.

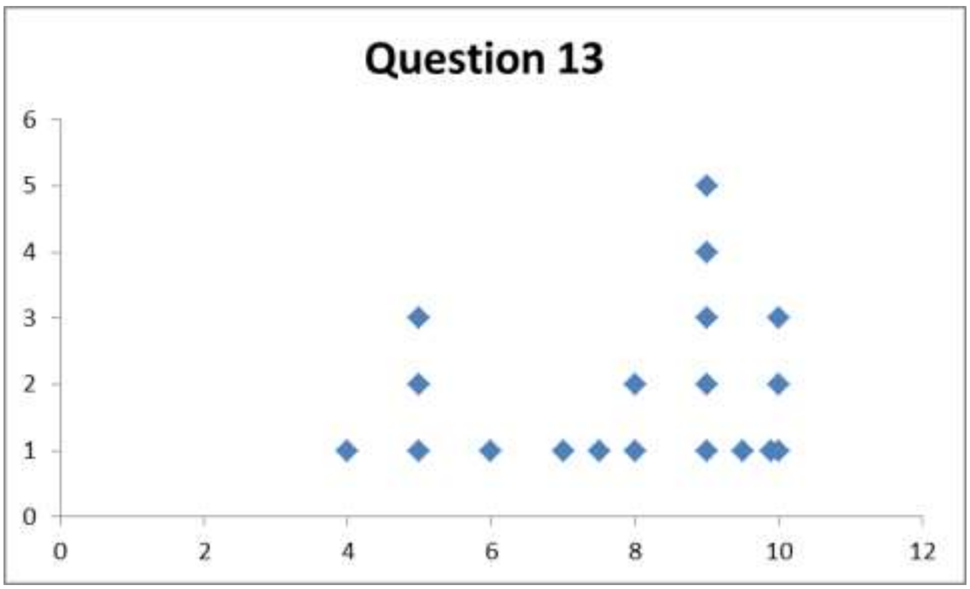


13) Write a grammar that generates the strings over $\{a, b\}$ starting with abb. (10 points)

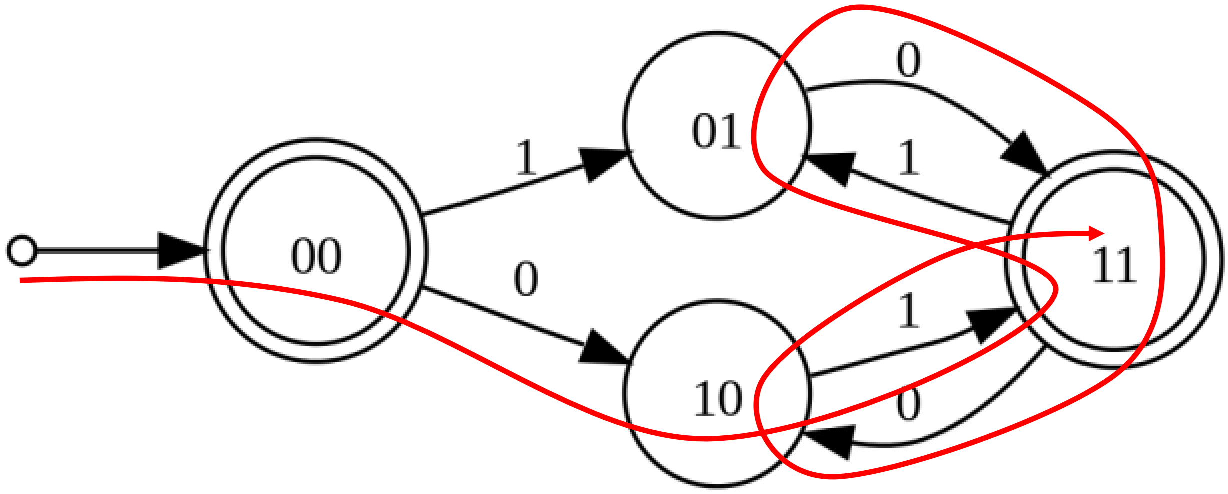
There are many correct answers. One such answer is:

$N = \{x, y\}$
$T = \{a, b\}$
$P = \{y \to abbx, x \to ax|bx|a|b|""\}$
$\sigma = y$

In this manner every string must start with "$abbx$" so that all generated strings must start with abb. Then x can be used to add any string to the end of that, even the empty string.



14) Below is a state transition diagram for a finite state automaton.



a) What are the accepting states? (2 points)

The two circled states: 00 and 11.

b) Is the input "011001" accepted? Clearly justify and/or illustrate your answer. (8 points)

Yes it is, as traced above the state transitions via 00 → 10 → 11 → 01 → 11 → 10 → 11, and the ending state, 11, is an accepting state.

Note that output is not shown in this state diagram, but from the states we can determine that the output would be "010101". Again, note that the last output is "1", which means accept.