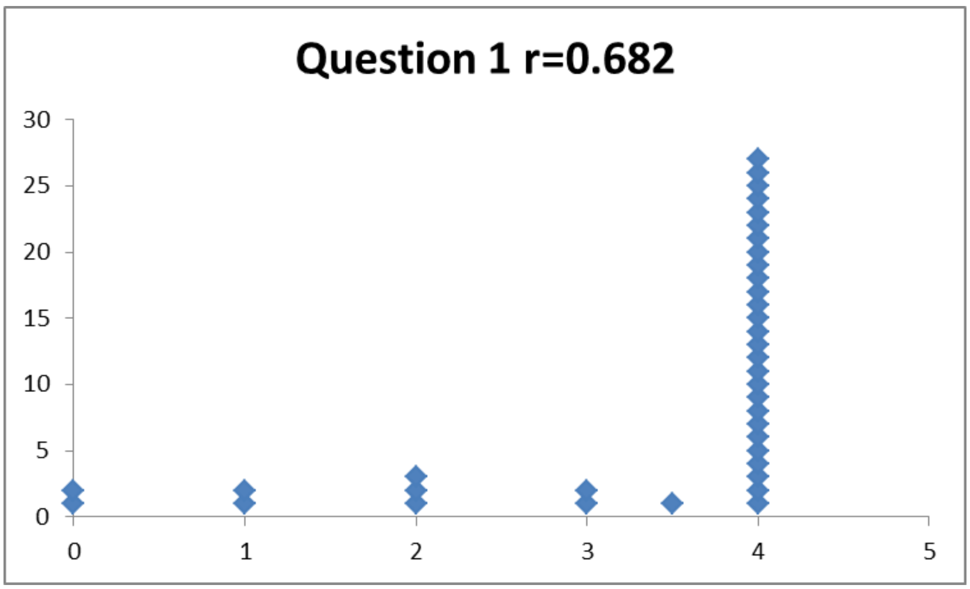
Name ____ Discrete II, 2/5/2016

Throughout this test, show all work and leave answers as meaningful expressions.

Use the following scenario for the next two problems. Suppose you have a bank account that earns 4% interest each year.

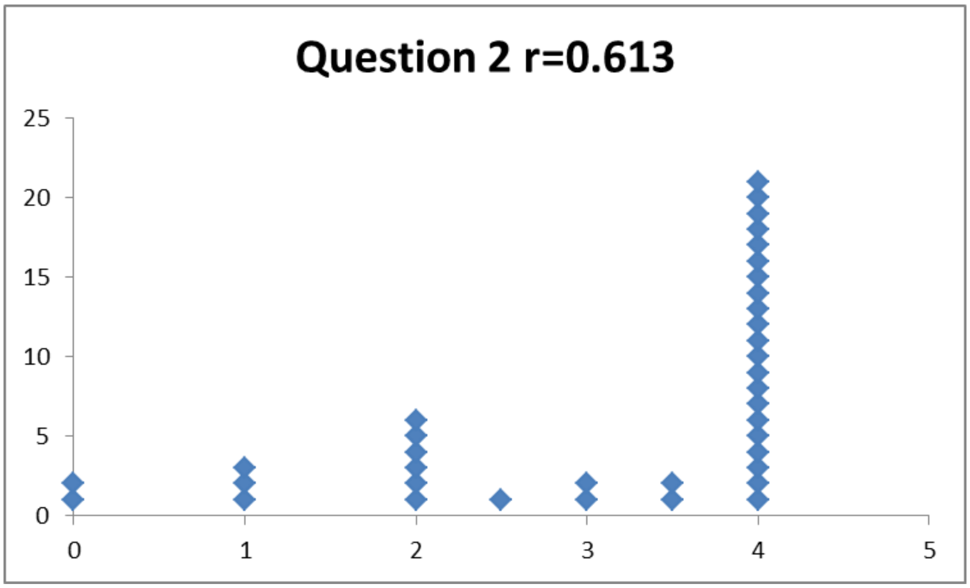
1) Write down the recurrence relation for the amount of money in the account after n years. (4 points)

$$a_n = 1.04a_{n-1}$$



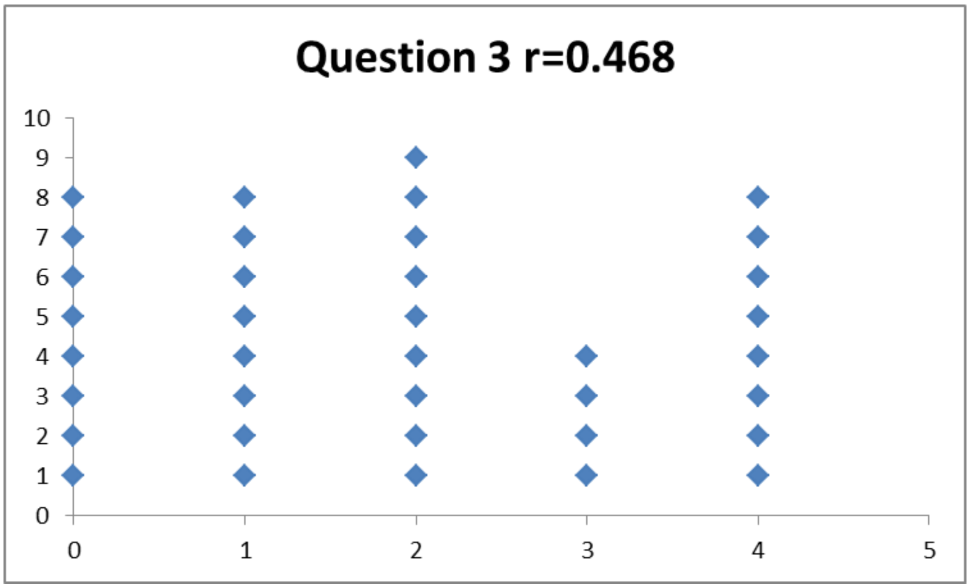
2) Now suppose you start with \$300 in the account. Find the value of the account after 2 years. (4 points)

$$a_2 = 300 \cdot 1.04^2$$



3) Still supposing that you start with \$300 in the account, find the value of the account after n years. (4 points)

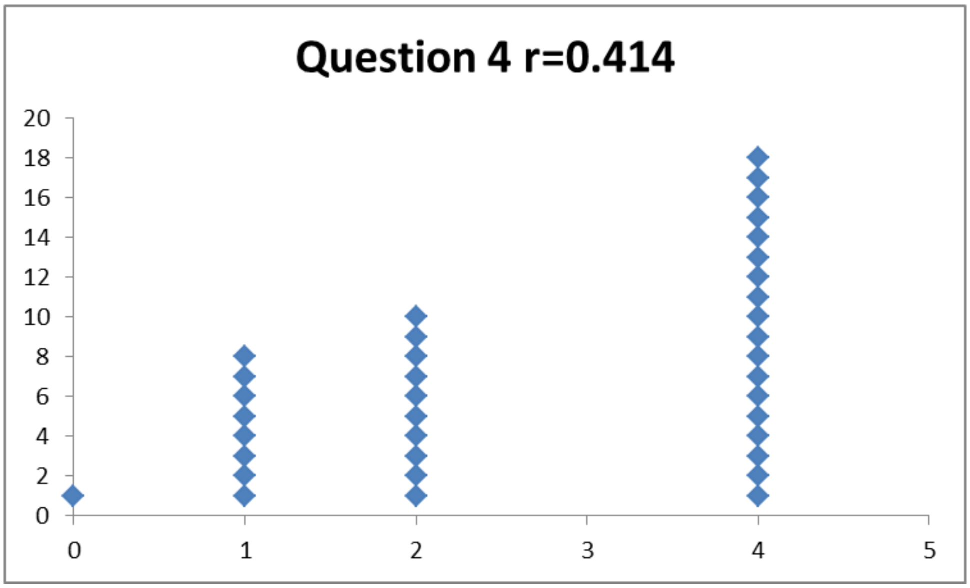
$$a_2 = 300 \cdot 1.04^n$$



4) A license plate will consist of four capital letters, followed by 4 digits. How many possible license plates are there? (4 points)

In a license plate we're allowed to repeat characters, and we need to specify each letter so this is a multiplication problem:

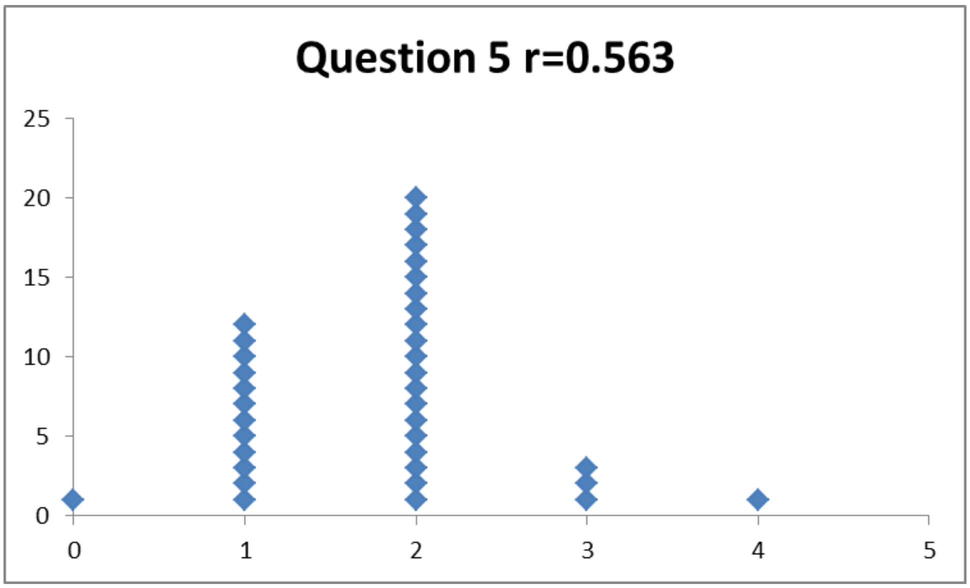
$$26^4 \cdot 10^4$$



5) A license plate will consist of four capital letters and 4 digits. (Unlike the previous question they might be mixed up so that AB2C4G57 is valid). How many possible license plates are there? (4 points)

Now we need to take each answer from the previous problem and mix them up. Of the 8 slots, we choose 4 of them to be letters. The rest will be numbers:

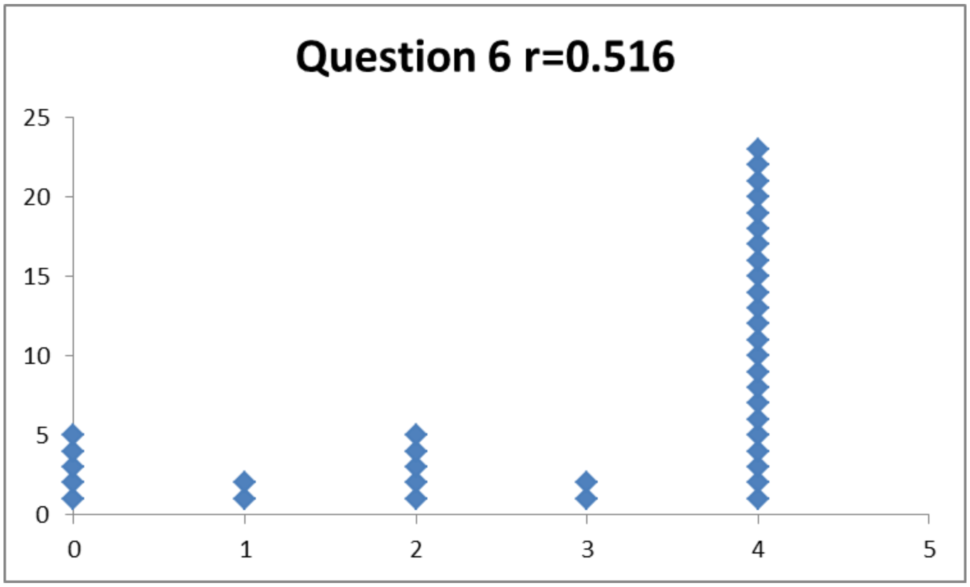
$$\binom{8}{4}26^4 10^4$$



6) A license plate will consist of four capital letters and 4 digits. How many possible license plates are there that do not consist of 4 letters followed by 4 digits? (4 points)

Here we want license plates that satisfy problem 5, but not 4. So subtract the two answers:

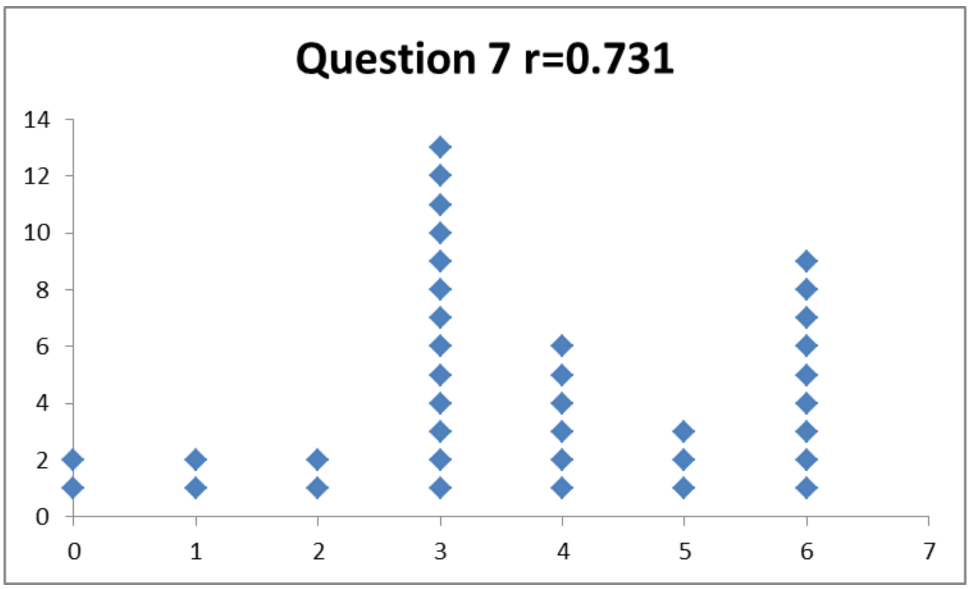
$$\binom{8}{4}26^4 10^4 - 26^4 10^4$$



Use the following scenario for the next **three** problems. A restaurant has 6 appetizers, 15 main dishes, and 8 desserts. You and a date are eating together. When sharing plates, you each take half without regard to who is served first.

7) How many ways can you order 3 appetizers? You're going to share. You don't mind if you have multiple of the same appetizer, because you just love all of them so much! (6 points)
You want 3 appetizers, it doesn't matter which order you get them, and you're doing it with repetition. So really you're wondering of the 6 appetizers available, how many of each you're going to have. That's stars and bars, because we're solving $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 3$. Now it's an easy problem:

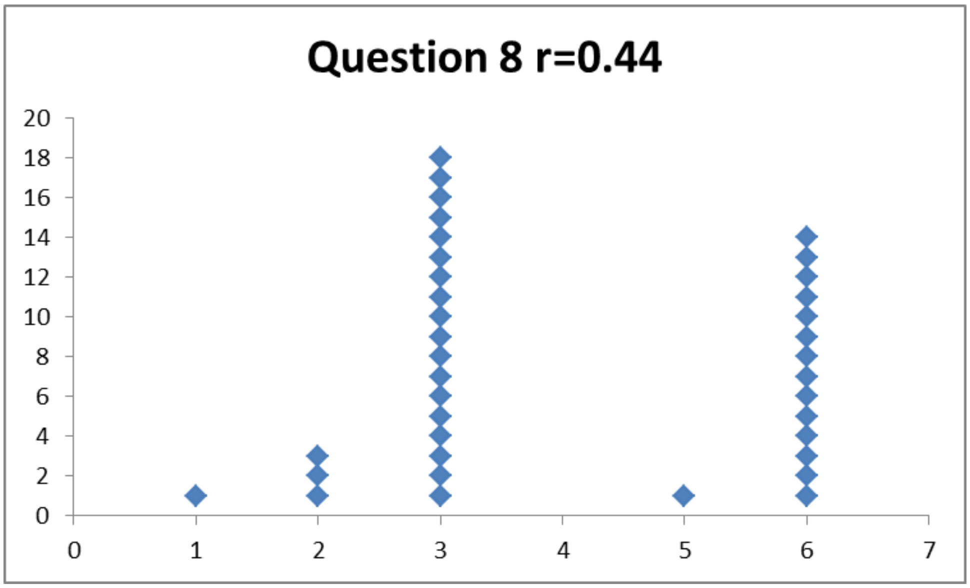
$$\binom{8}{5}$$



8) How many ways can you order 2 main dishes? You're not going to share. (6 points)

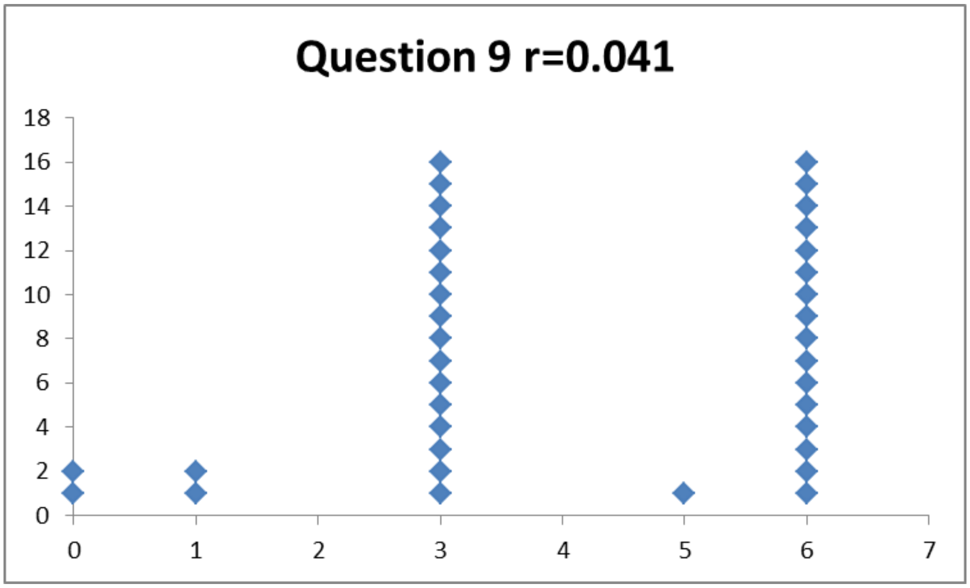
The fact that you're not going to share makes this problem really easy: you have 15 choices, so does your date:

$$15^2$$



9) How many ways can you order 2 different desserts? You're going to share. (6 points)
You want different desserts here, so there's no repetition. But you're going to share, so the order doesn't matter. This is a combination:

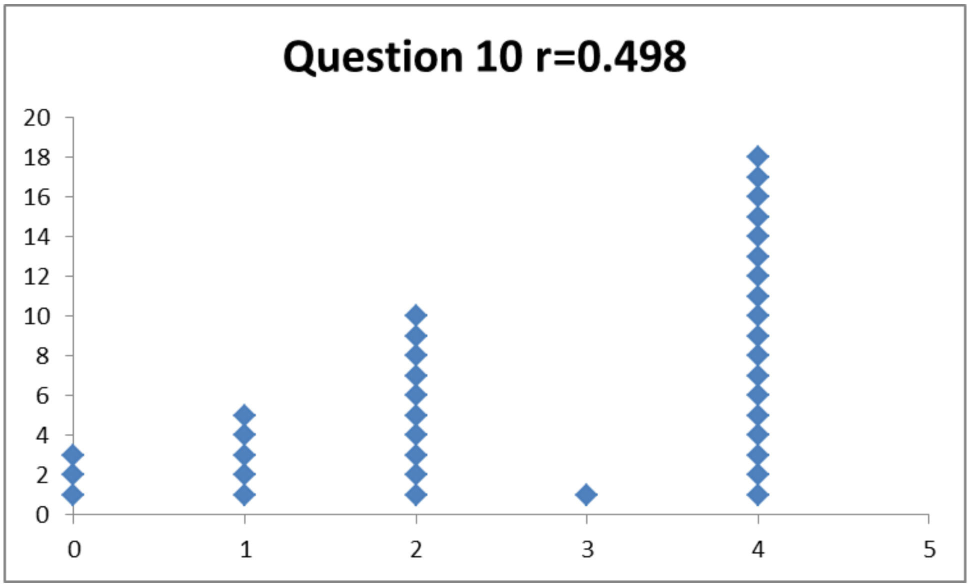
$$\binom{8}{2}$$



10) An ice cream parlor has 32 flavors of ice cream. Seven customers walk in and each want one scoop of ice cream. How many ways can this be done? (4 points)

Each person can get whatever they want, and we can tell the difference between who gets what. So this is a straight multiplication problem:

$$32^7$$



11) You have three decks of cards and you draw one card from each deck. How many ways can you end up with 3 red cards? (4 points)

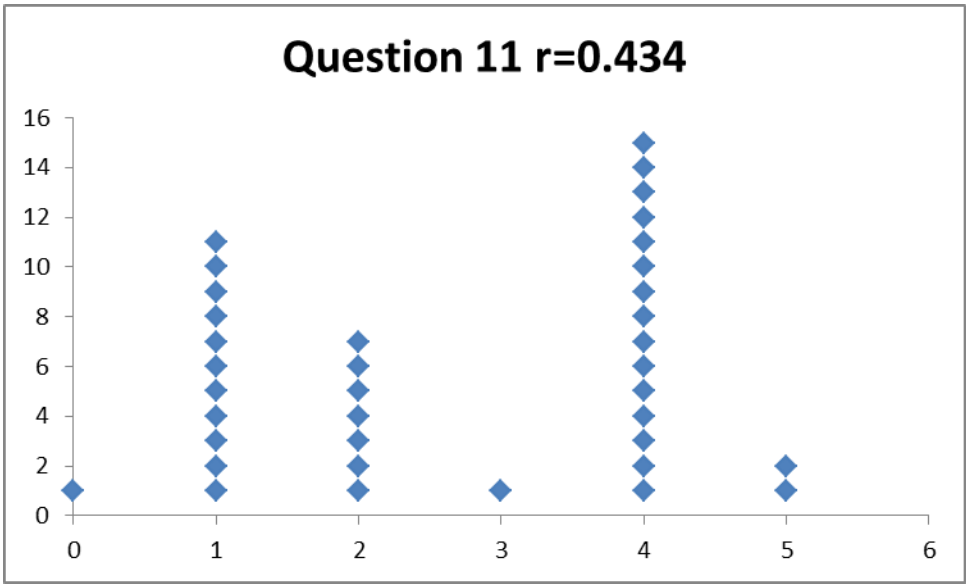
Each deck gives us 26 ways to get a red card. So we get a 26 for each deck:

$$26^3$$

Oh... wait... I didn't realize there's another way to interpret this question. Above I was assuming you can tell the difference between the cards from different decks. What if you can't? Then it's a stars and bars problem:

$$\binom{3+25}{25}$$

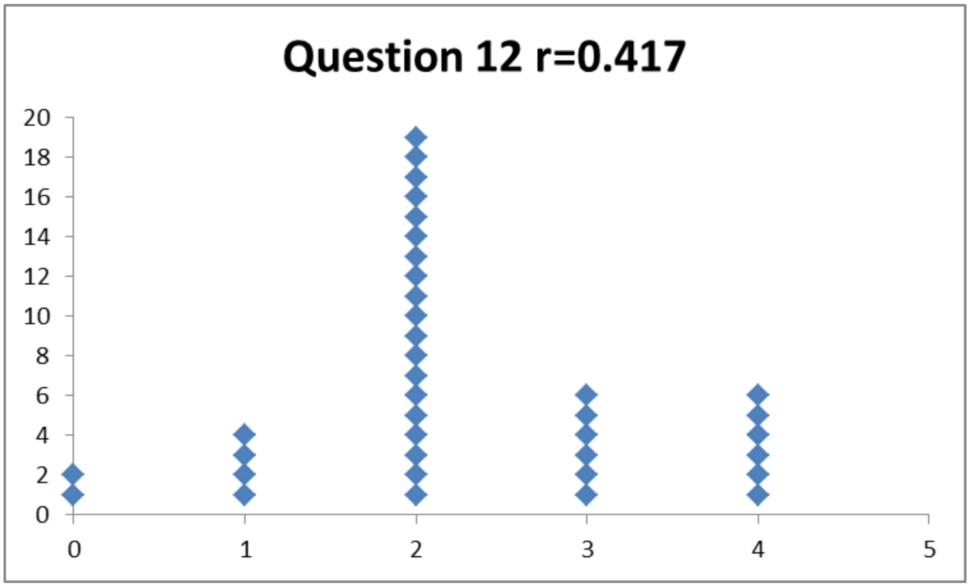
Either answer would have been accepted, although nobody tried to do it the latter method.



12) You draw 3 cards from a standard deck of cards. How many ways can you end up with a 3-of-a-kind? An example of a 3 of a kind is 3 jacks. (4 points)

Luckily we are only taking 3 cards, so we don't need to worry about anything else. Of the 13 card types, we want to choose one of them to be our type. Then within that type there are 4 of those cards, and we choose 3 of them:

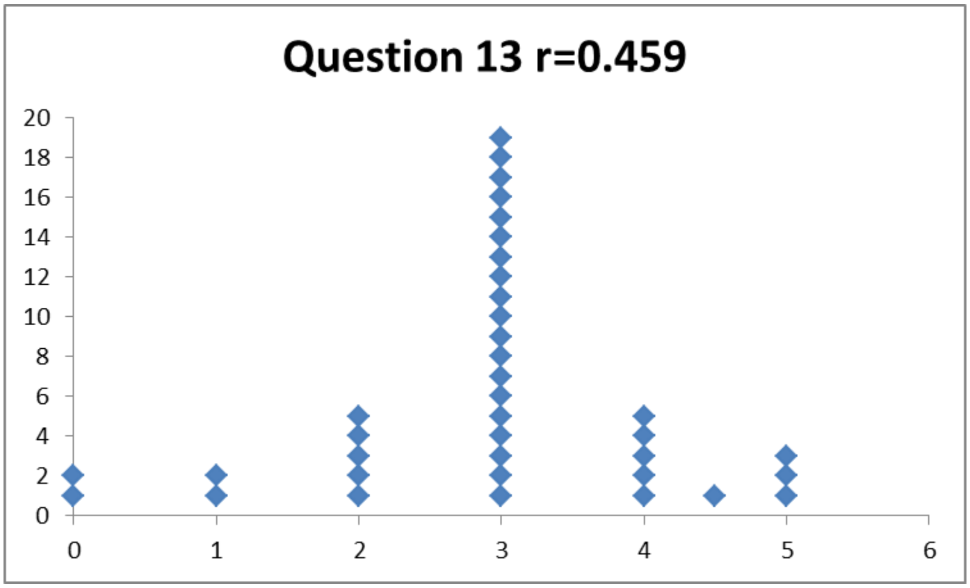
$$\binom{13}{1}\binom{4}{3}$$



13) You draw 4 cards from a standard deck of cards. How many ways can you end up with two different pairs? An example of two pairs is 2 jacks and 2 queens. (6 points)

This question is a little tougher because there are two different types. So we need to select two types (order doesn't matter). Then select the cards for both:

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}$$



Consider the recurrence relation below for the next **three** questions.

$$a_n = 8a_{n-1} - 16a_{n-2}$$

$$a_0 = 5; a_1 = 36$$

14) Find a solution to the recurrence relation. (6 points)

First we set up the characteristic equation:

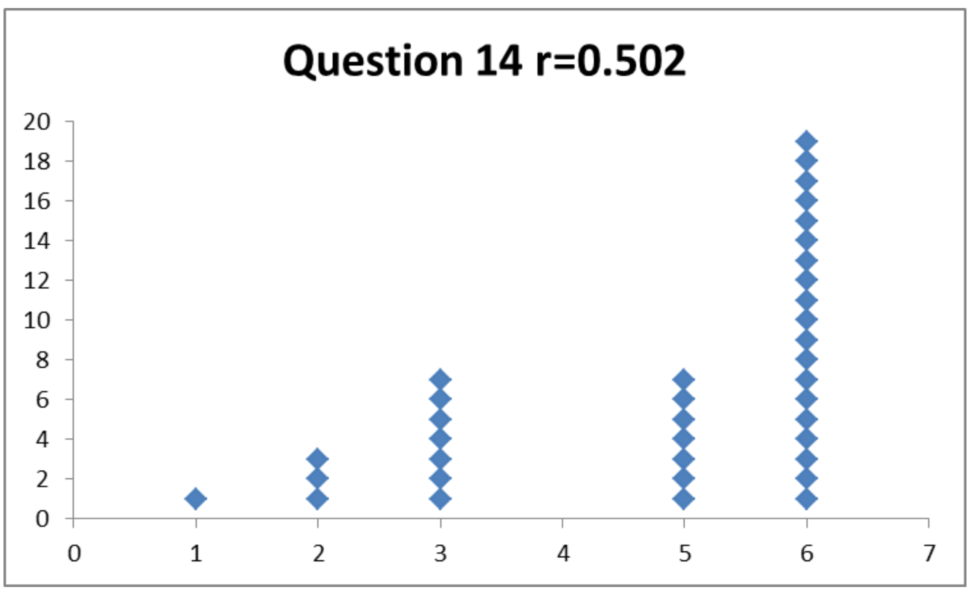
$$x^2 = 8x - 16$$

Next we move everything to the left side of the equation and then factor:

$$(x-4)^2 = 0$$

This has the solution repeated 4. We use that to construct two solutions to the recurrence relation: $a_n = 4^n$ and $a_n = n \cdot 4^n$. We choose one of these to present as a solution to the recurrence relation:

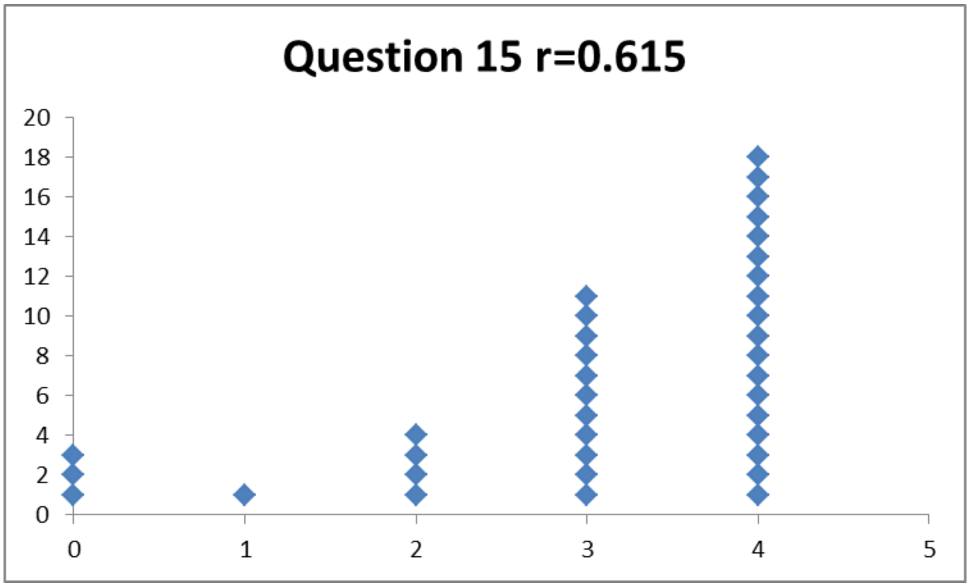
$$a_n = 4^n$$



15) Find the general solution to the recurrence relation. (4 points)

The general solution is the arbitrary linear combination of the two solutions we found:

$$a_n = c \cdot 4^n + d \cdot n \cdot 4^n$$



16) Find the particular solution to the recurrence relation with the given initial conditions.
(4 points)

Now using the initial conditions we set up a system of two variables and two linear equations:

$$c \cdot 4^0 + d \cdot 0 \cdot 4^0 = 5$$

$$c \cdot 4^1 + d \cdot 1 \cdot 4^1 = 36$$

Simplify these equations to obtain:

$$c = 5$$

$$c \cdot 4 + d \cdot 4 = 36$$

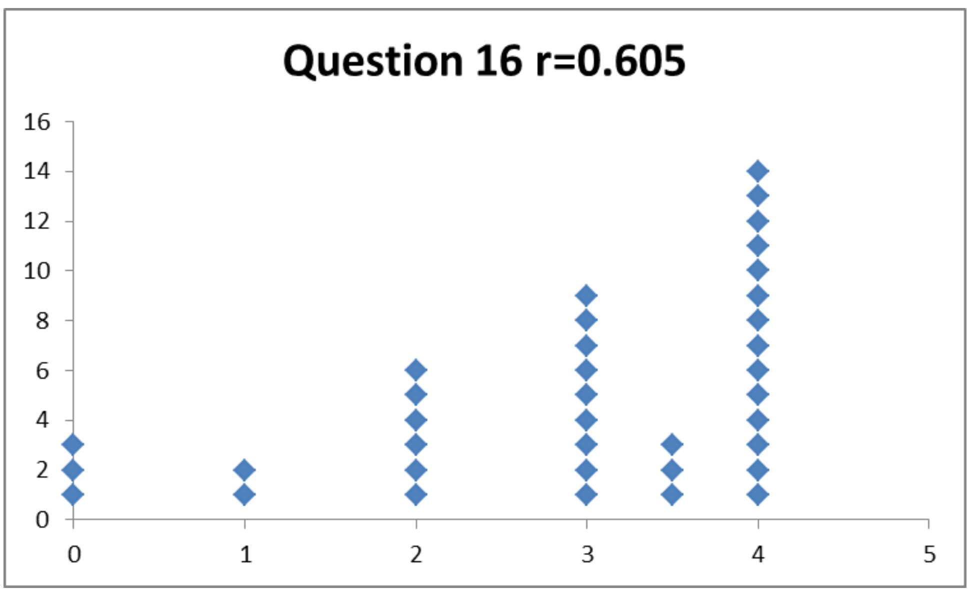
That's awesome, the first equation gave us c! Plug that in to the second equation to get:

$$20 + 4d = 36$$

$$d = 4$$

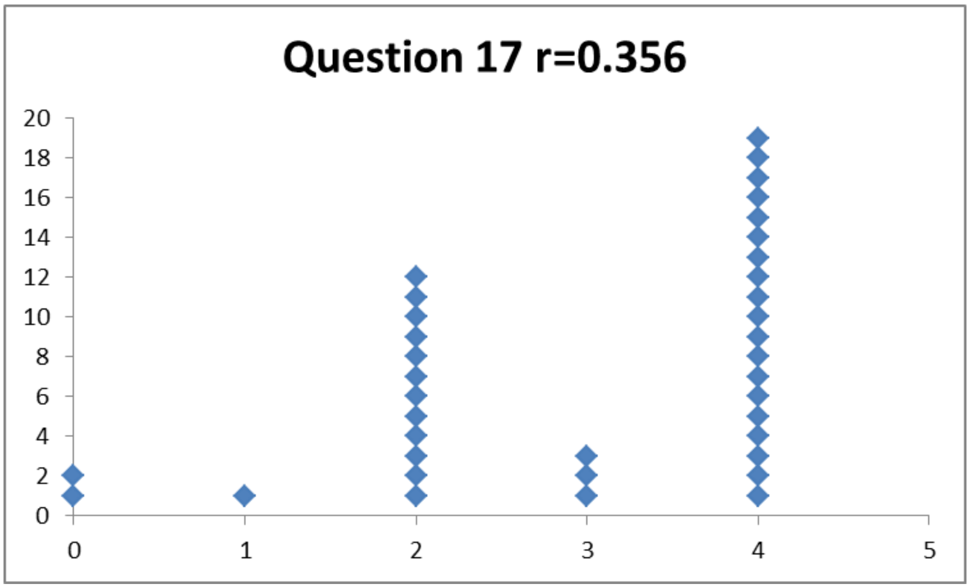
Hence the particular solution given the initial conditions is

$$a_n = 5 \cdot 4^n + 4 \cdot n \cdot 4^n$$



17) How many numbers are there between 111 and 400, including both 111 and 400? (4 points)
This problem looks really easy at first, but it's a little tricky. First let's count the numbers between 4 and 6. What did you do? Take 6-4=2? We miss one of the end points like that, because {4,5,6} actually contains 3 numbers. Similarly, here we see that there are:

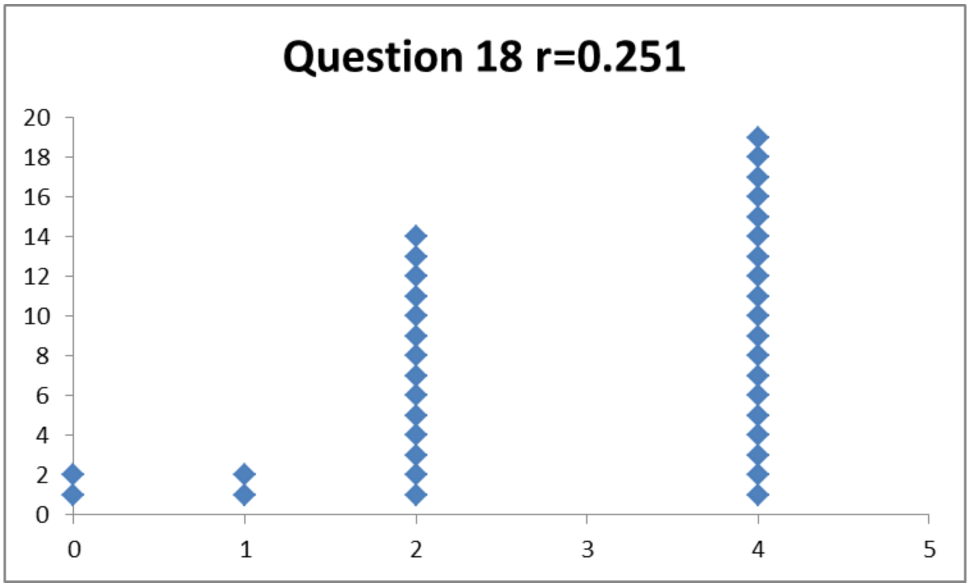
$$400 - 111 + 1 = 290 \text{ numbers}$$



18) How many numbers between 1 and 1200, including both, are divisible by either 5 or 10? (4 points)

This is a problem that looks harder than it actually is. First note that any number divisible by 10 is also divisible by 5 – and 1200 is divisible by both. So we really only need to worry about 5:

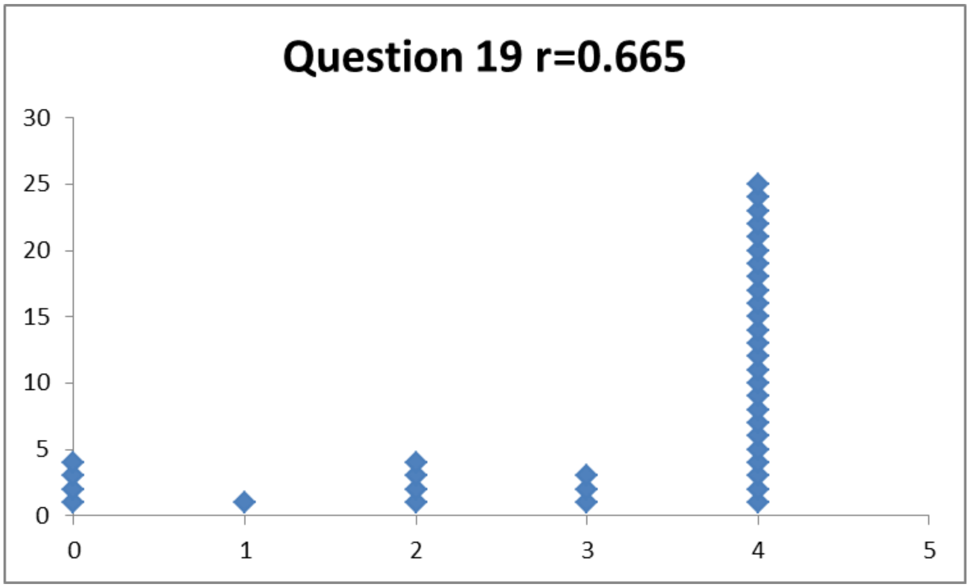
$$\frac{1200}{5}$$



19) How many nonnegative integer solutions are there to the equation below? (4 points)

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 33$$

$$\binom{33+6-1}{6-1} = \binom{33+6-1}{33}$$



20) How many nonnegative integer solutions are there to the equation below, given the constraint that $x_2 \geq 8$? (4 points)

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 33$$

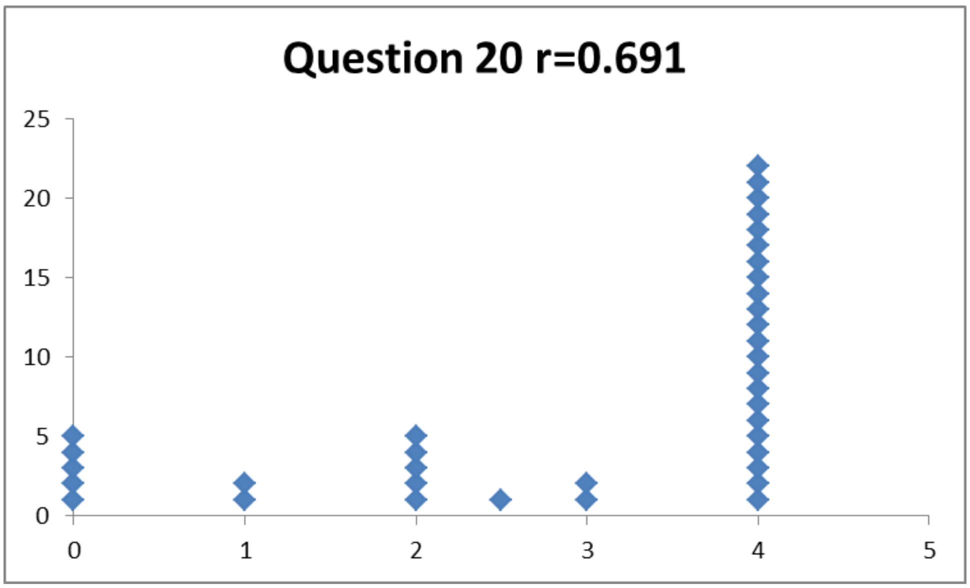
This one takes some work because of the constraint $x_2 \geq 8$. We make the change of variables $x_2' = x_2 - 8$. Plugging this into the equation we obtain:

$$x_1 + x_2' + x_3 + x_4 + x_5 + x_6 + 8 = 33$$

$$x_1 + x_2' + x_3 + x_4 + x_5 + x_6 = 25$$

We can now solve this using stars and bars:

$$\binom{25+6-1}{6-1} = \binom{25+6-1}{33}$$



21) How many nonnegative integer solutions are there to the equation below, given the constraint that $x_2 \leq 10$? (4 points)

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 33$$

This one takes even more work, because a simple change of variables won't work. (Try it, you'll see that the inequality is going the wrong direction.) Instead we'll use inclusion-exclusion to find "all" solutions and remove the "bad" solutions. That will yield just the "good" solutions. For "all" solutions, ignore the constraint and just solve $x_1+x_2+x_3+x_4+x_5+x_6 = 33$ with each variable a nonnegative integer:

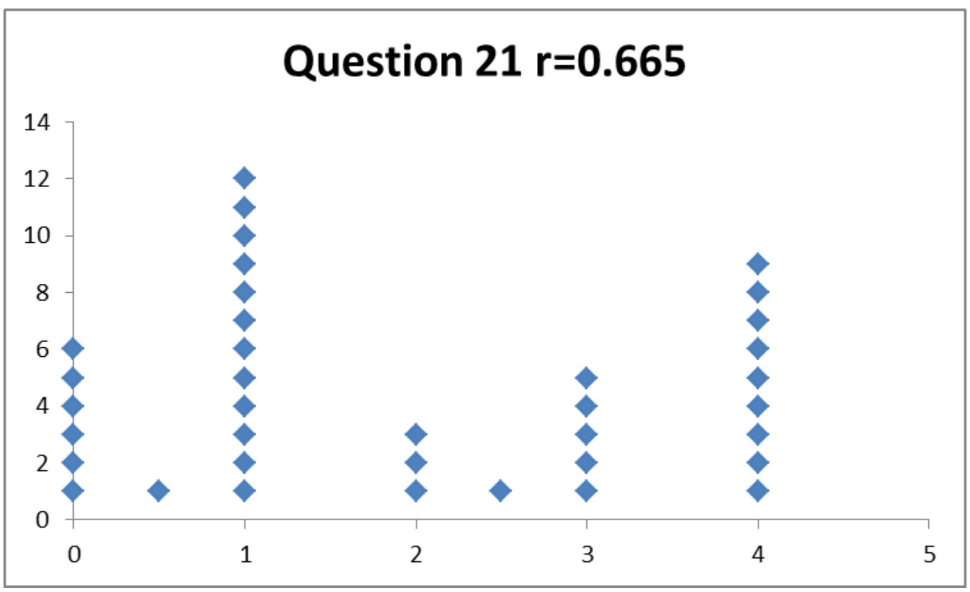
$$\binom{38}{5} = \binom{38}{33}$$

Now a "bad" solution has $x_2 \geq 11$, so we make the change of variables $x_2' = x_2 - 11$ that result in the equation $x_1 + x_2' + x_3 + x_4 + x_5 + x_6 = 22$. Using stars and bars we see that we get this many solutions:

$$\binom{27}{5} = \binom{27}{22}$$

Lastly we subtract the two to get just the "good" solutions:

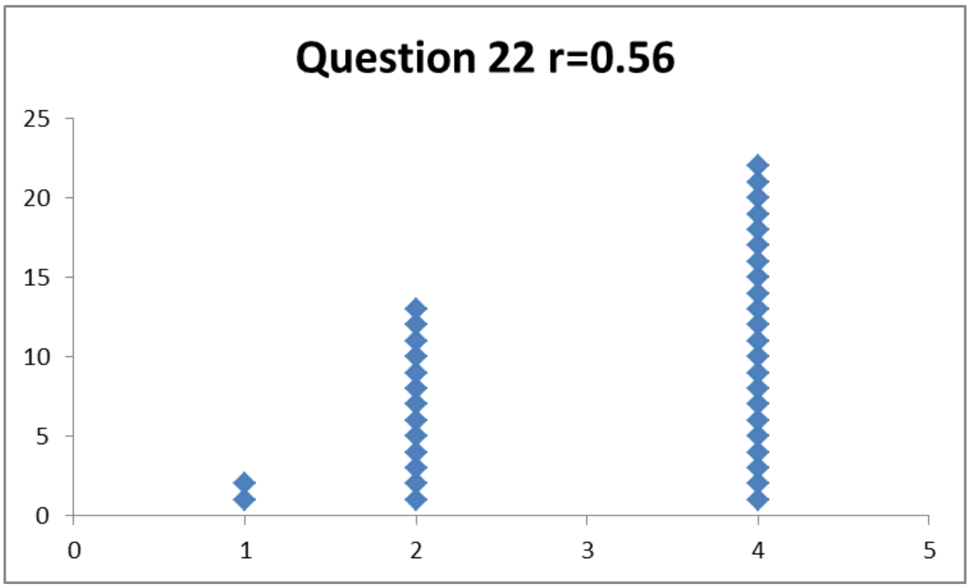
$$\binom{38}{5} - \binom{27}{5}$$



22) A Blockbuster movie rental store is having a clearance sale to liquidate their inventory before closing. They have 350 different DVDs to sell. How many ways can they sell the DVDs to 7 customers, such that each gets 50 DVDs? (4 points)

We can tell the difference between the DVDs and between the customers, but not the order in which we give them to any given customer. Hence this is a series of combinations - that is, a multicombination:

$$\binom{350}{50\ 50\ 50\ 50\ 50\ 50\ 50} = \binom{350}{50}\binom{300}{50}\binom{250}{50}\binom{200}{50}\binom{150}{50}\binom{100}{50}\binom{50}{50}$$



23) Calculate $\binom{7}{2}$. (Do the arithmetic until you get a single number as your answer) (4 points)

To calculate a combination, you can use the formula $\binom{7}{2} = \frac{7!}{2!5!}$ and go from there, but I prefer to think about it in terms of selections: in the numerator you select 2 things with order, and in the denominator you divide by the symmetries between rearranging two identical objects:

$$\binom{7}{2} = \frac{7 \cdot 6}{2 \cdot 1} = 7 \cdot 3 = 21$$