Name ______________________________ Discrete II, 2/29/2016, Test 2

1) List the first seven 3-combinations from the set $\{a, b, c, d, e\}$ in lexicographic order. (6 points)

abc

abd

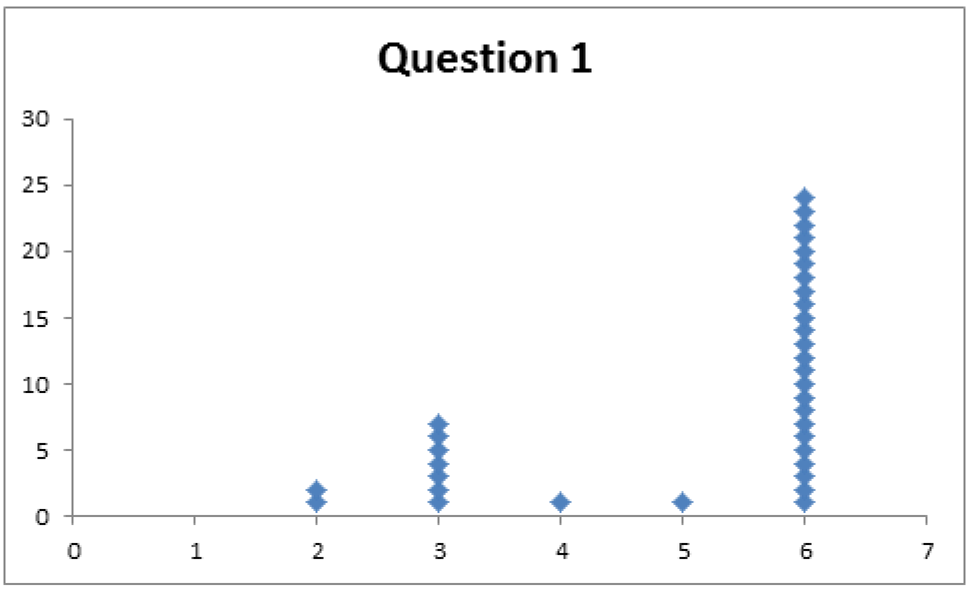
abe

acd

ace

ade

bcd



2) List all permutations from the set $\{4, 6, 7\}$ in lexicographic order. (6 points)

467

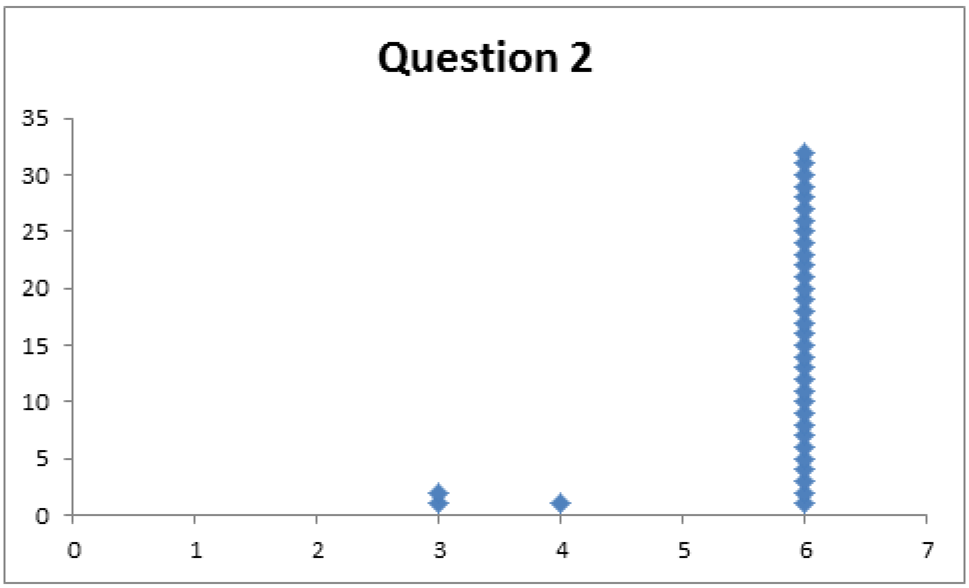
476

647

674

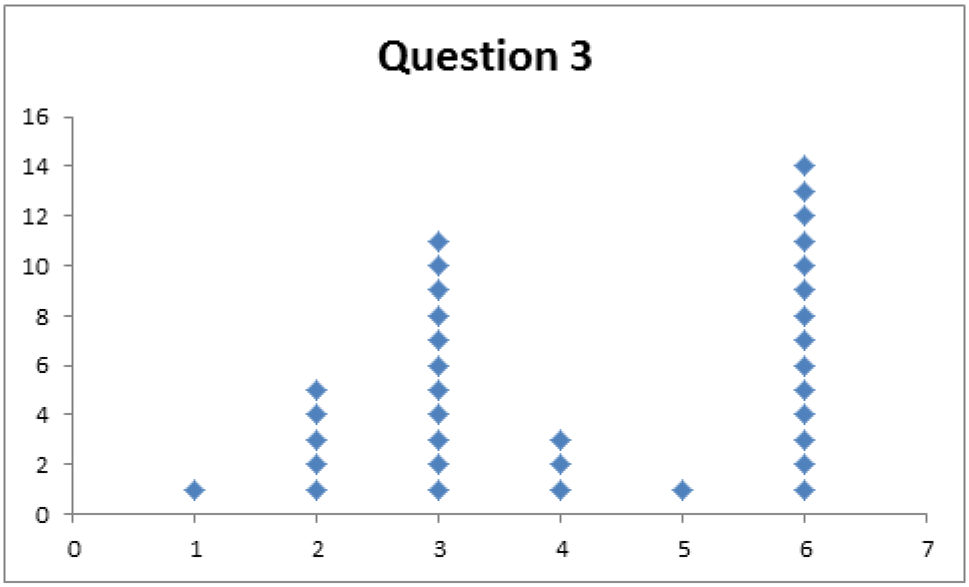
746

764



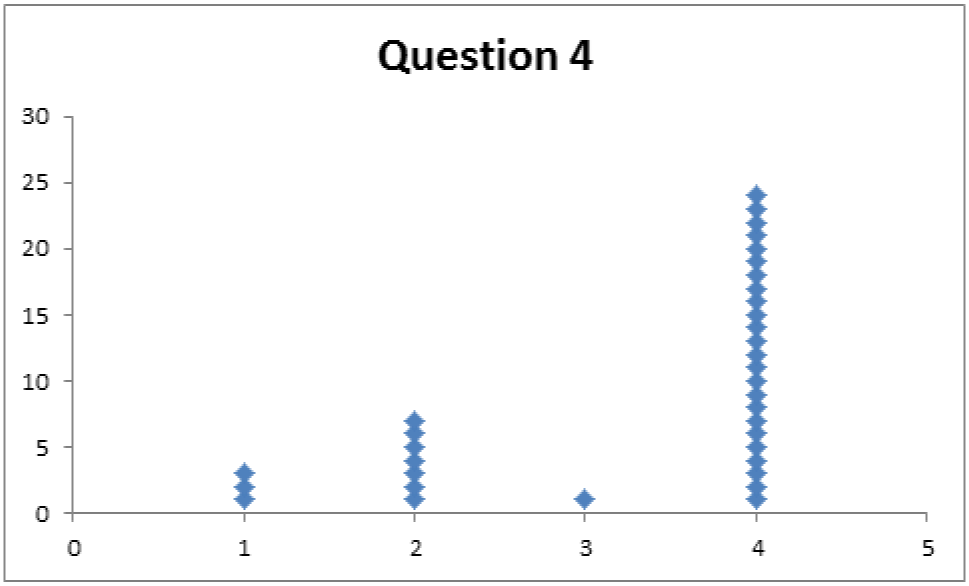
3) You flip a coin 10 times. What is the probability of getting three heads and the rest tails? (6 points)

$$\frac{\binom{10}{3}\binom{7}{7}}{2^{10}} = \frac{\binom{10}{3}}{2^{10}}$$



4) A card is selected at random from a standard deck of 52 cards. What is the probability that it is a five? (4 points)

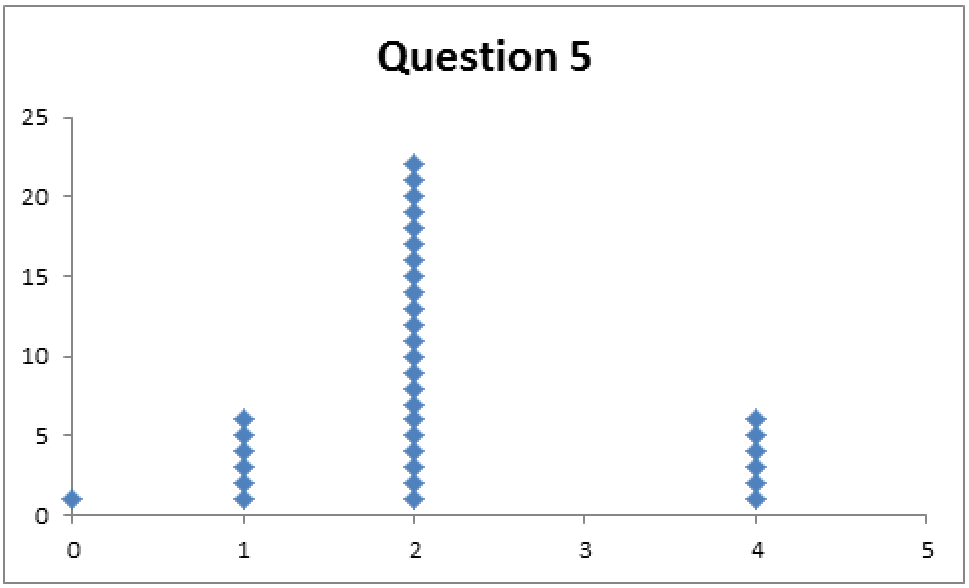
$$\frac{4}{52}$$



5) A standard deck of 52 cards is sitting on a table. You draw 2 cards. What is the probability that the second card you draw is a heart? (4 points)

A lot of people over thought this question. Note that you don't know anything about the first card, so this question would be equivalent to asking what the probability is for the 17th card. In fact, every single card has the same chance of being a heart. Hence the probability is:

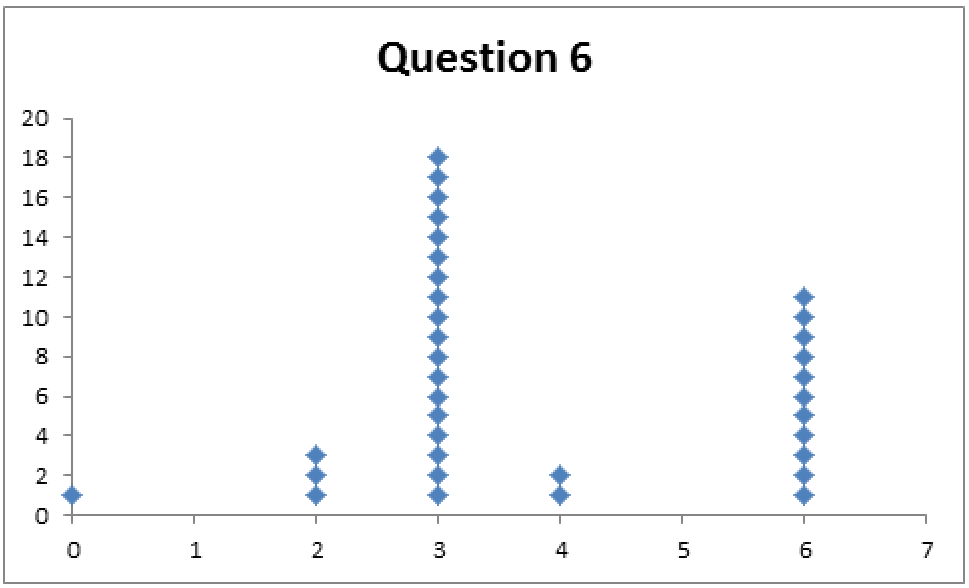
$$\frac{13}{52}$$



6) Two standard dice are rolled. What is the probability that the two numbers sum to 10? (6 points)

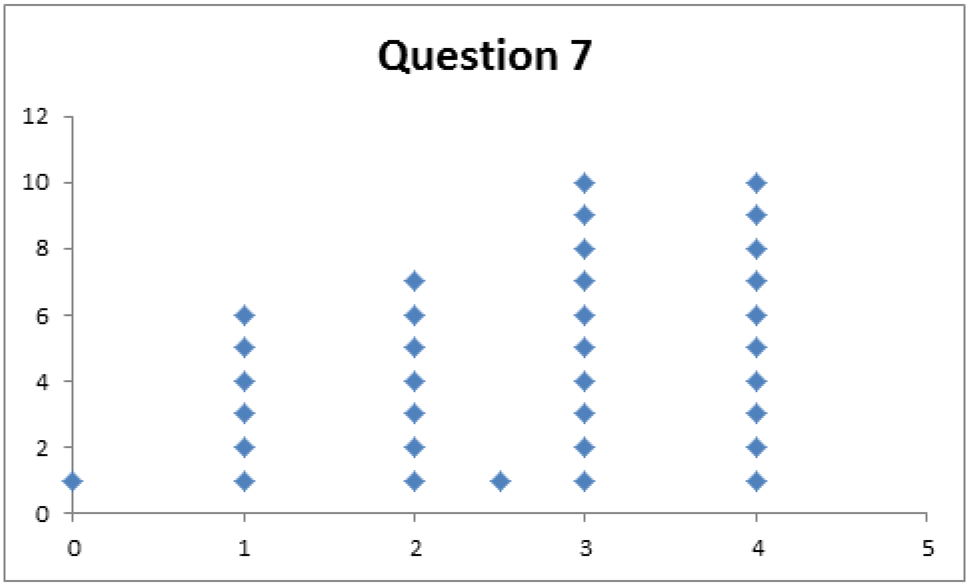
The denominator here is 6^2 because there are two dice, and so between them there are 6^2 different possibilities. For the numerator we need to figure out how many ways we can get a 10. These are: 4+6; 5+5 and 6+4. Note that when we get doubles there is only one way to do that: $5+5$. However, for the others there are two different events that look similar. Hence the total probability is:

$$\frac{3}{36}$$



7) Write down $(a+b)^{38}$ as a summation of each of the 39 terms. (4 points)

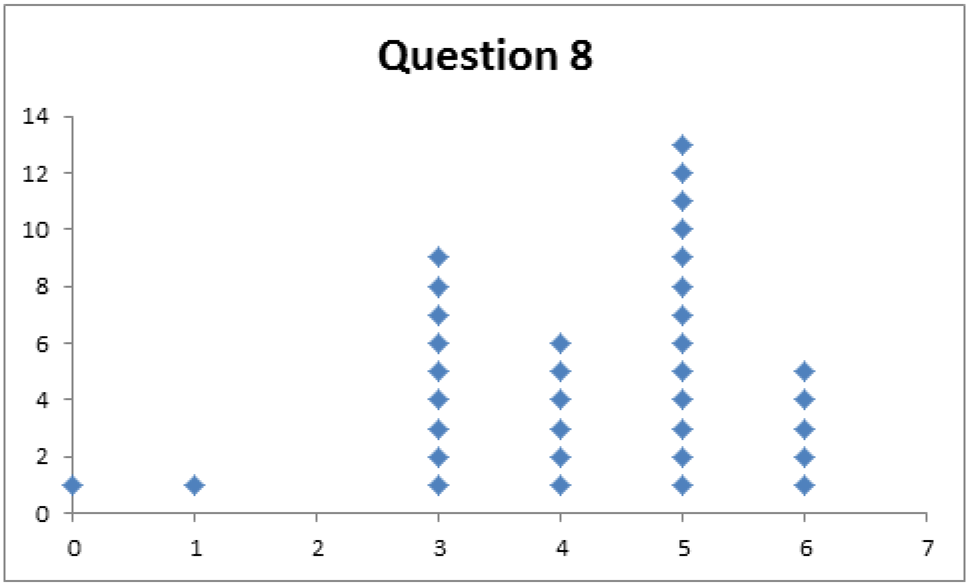
$$(a+b)^{38} = \sum_{n+m=38} \binom{38}{n} a^n b^m = \sum_{n=0}^{38} \binom{38}{n} a^n b^{38-n}$$



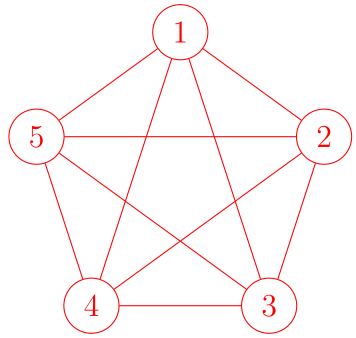
8) What is the term containing x^{15} in $(x+2y)^{50}$? (6 points)

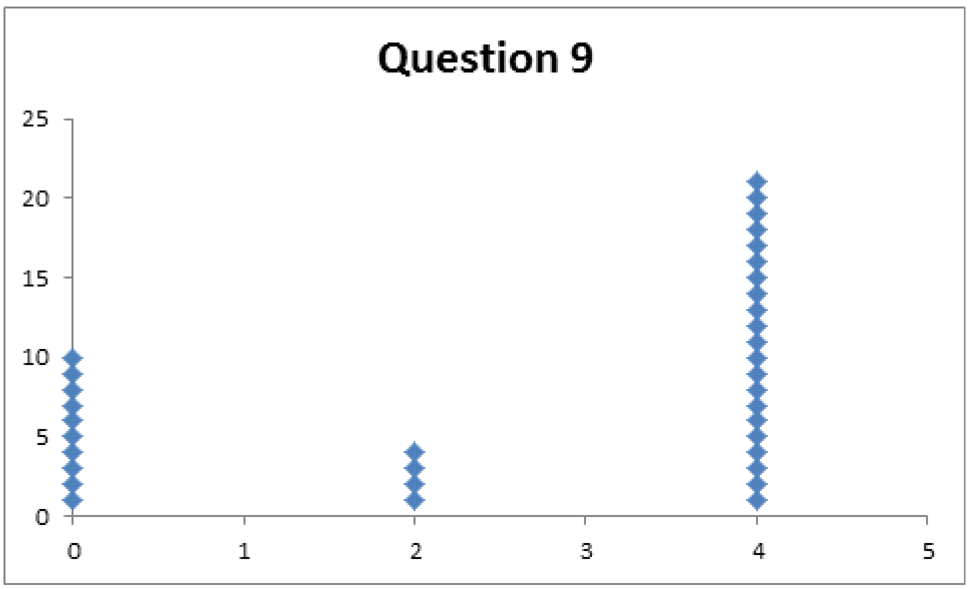
First note that the term containing x^{15} actually contains x^{15}. Using the binomial formula, given in the previous problem, we take just the term with the correct index:

$$\binom{50}{15}(2y)^{35}x^{15} = \binom{50}{15}2^{35}y^{35}x^{15}$$

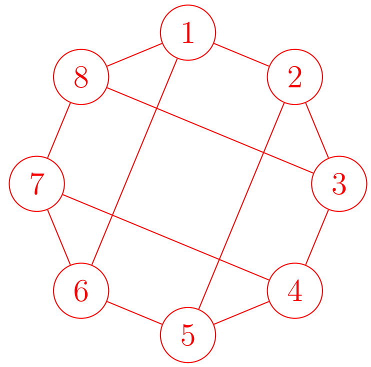


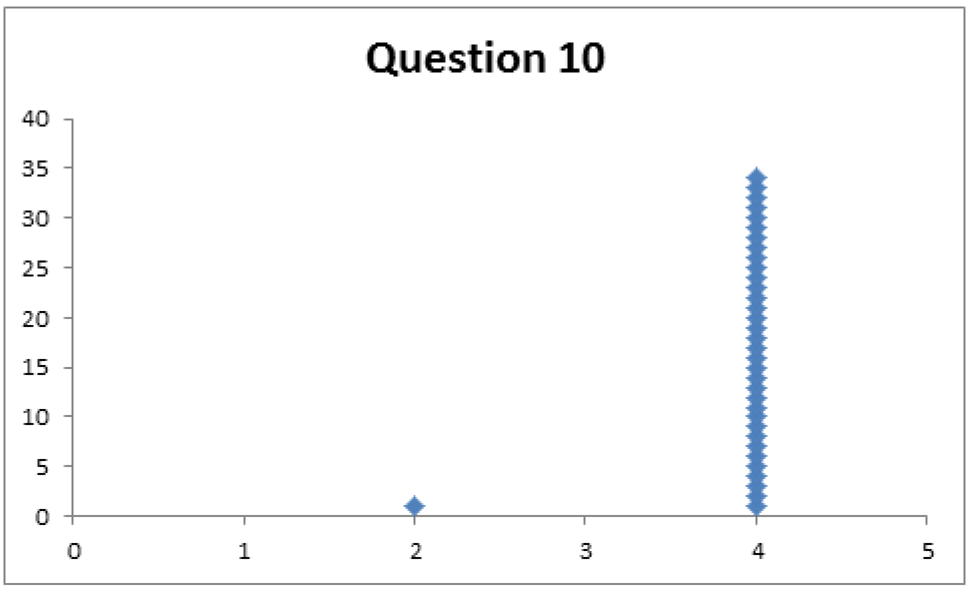
9) Draw an example of a complete graph with 5 vertices (4 points)



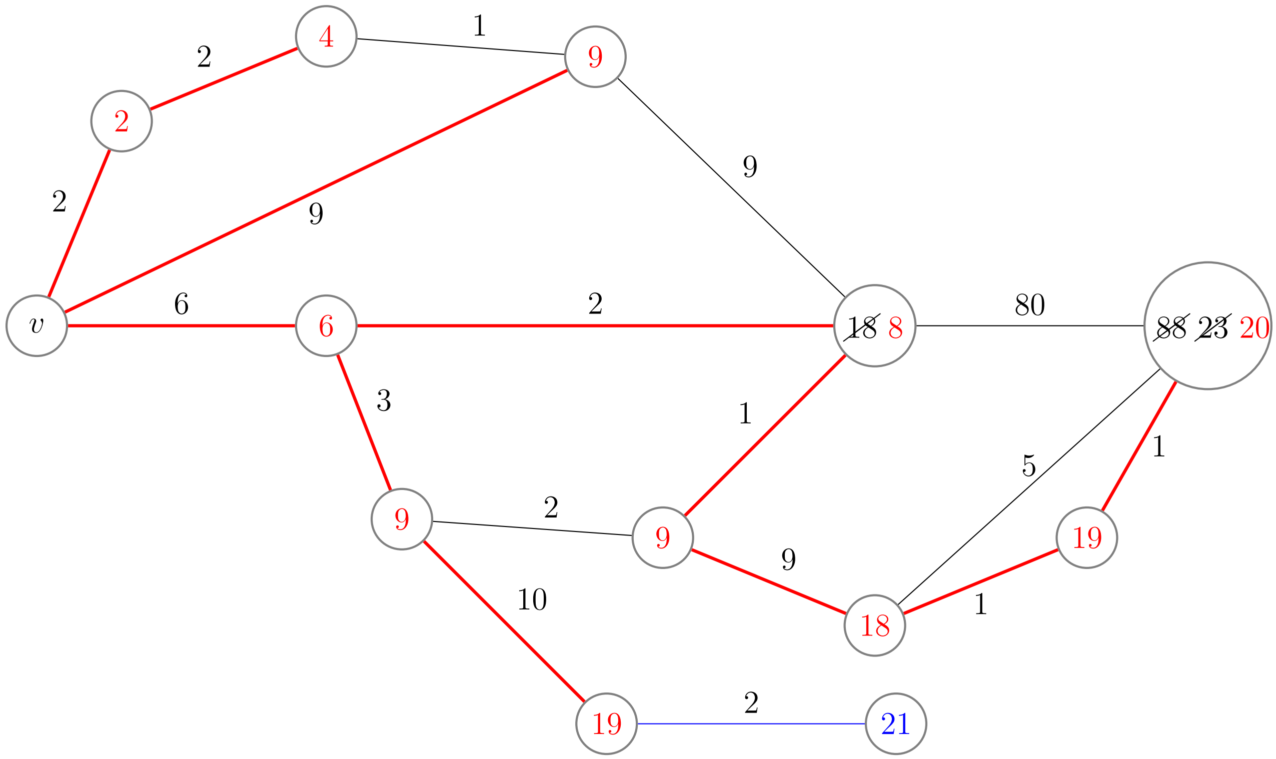


10) Sketch a graph with 8 vertices in which each vertex has degree 3. (4 points)

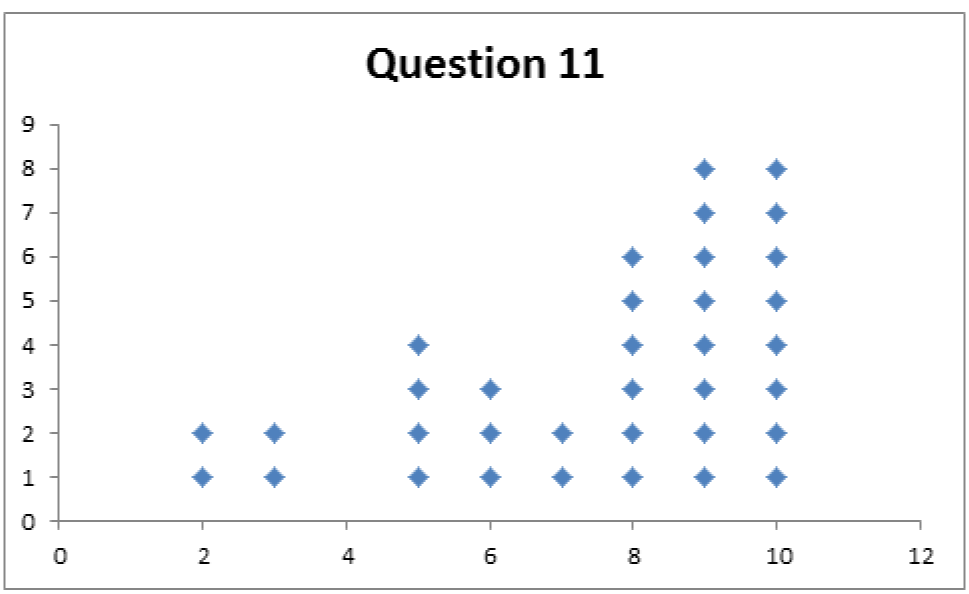




11) On the graph below, use Dijsktra's algorithm to find the shortest path from v to w. Please effectively illustrate how the algorithm works. (10 points)

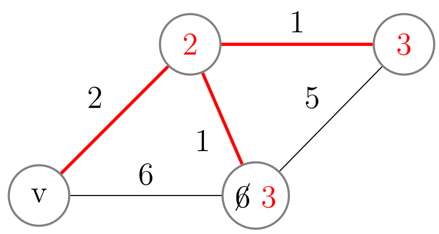


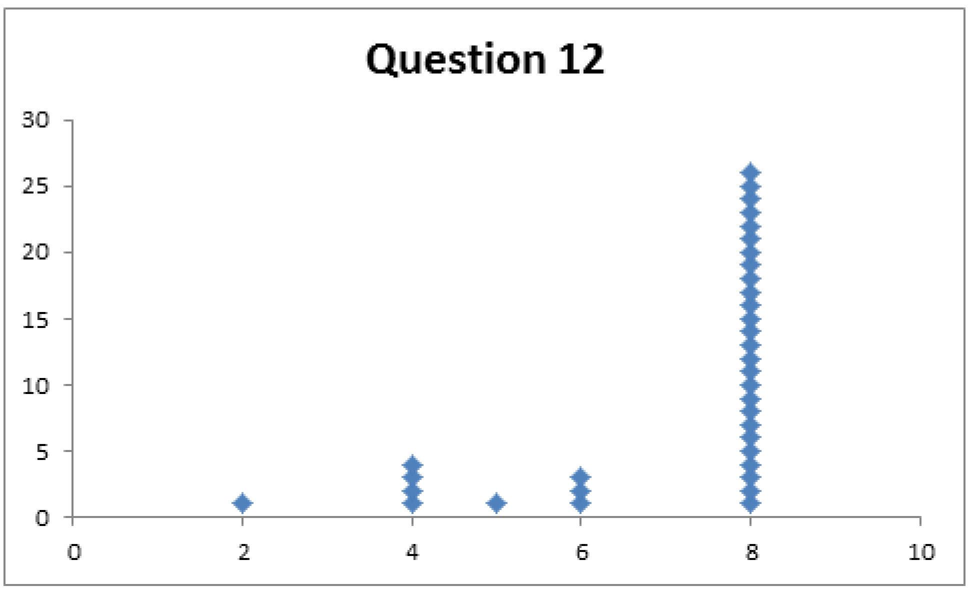
Legend: Thick red edges are optimal routes.
Blue edges are best-known routes, but aren't guaranteed to be optimal.
Crossed off numbers were best known at one point, but have since been replaced



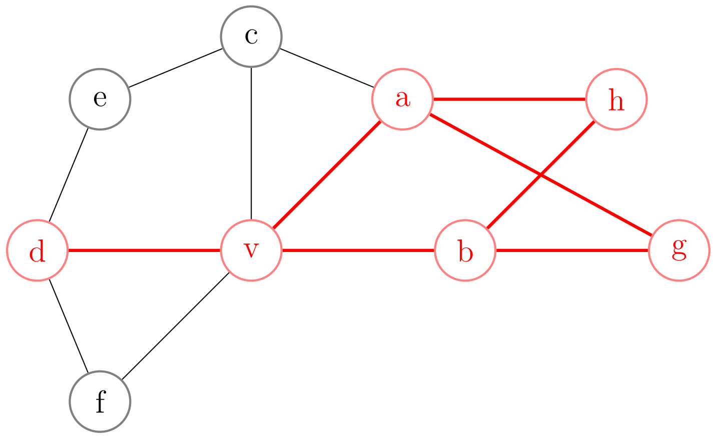
12) Below is a graph (actually 8 drawings of it). You can run Dijkstra's algorithm to find the shortest path from v to w - do so, using each new graph to illustrate one new step. You might not need all 8 copies of the graph. (8 points)

Below is the final graph according to Dijkstra's algorithm; presumably you gave it one step at a time on each graph so that it was clear how you were running the algorithm.

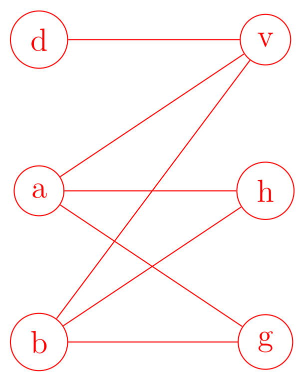


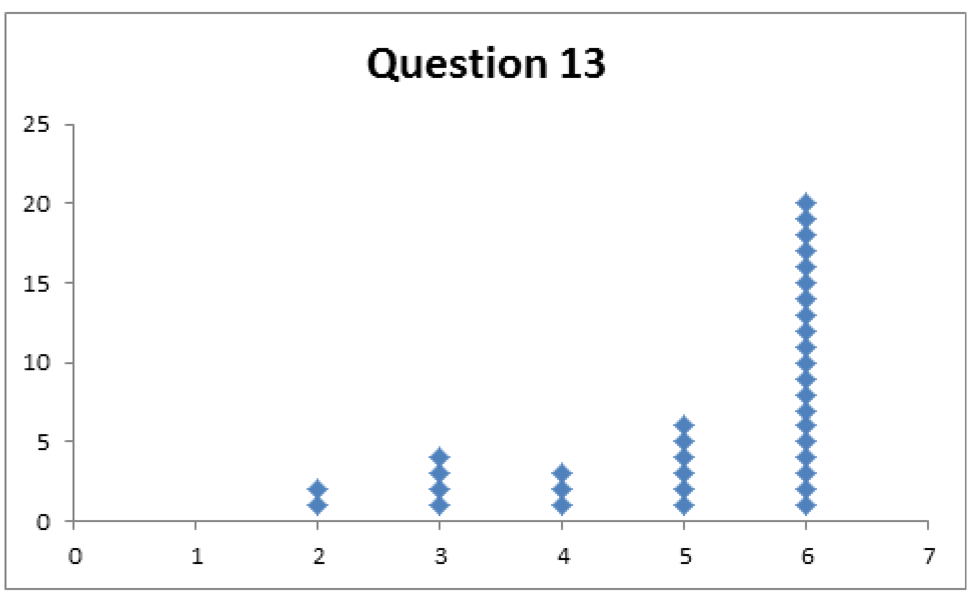


13) In the graph below, identify a bipartite subgraph. Illustrate why the subgraph you chose is bipartite. (6 points)



There are many solutions here as there are many bipartite subgraphs. Above we shaded one of them. Below is that graph redrawn as an obvious bipartite graph.

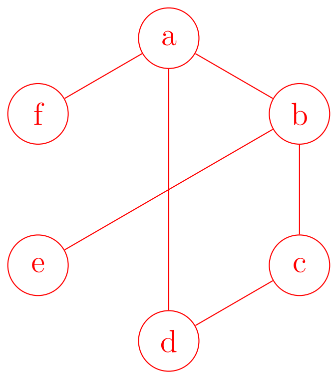


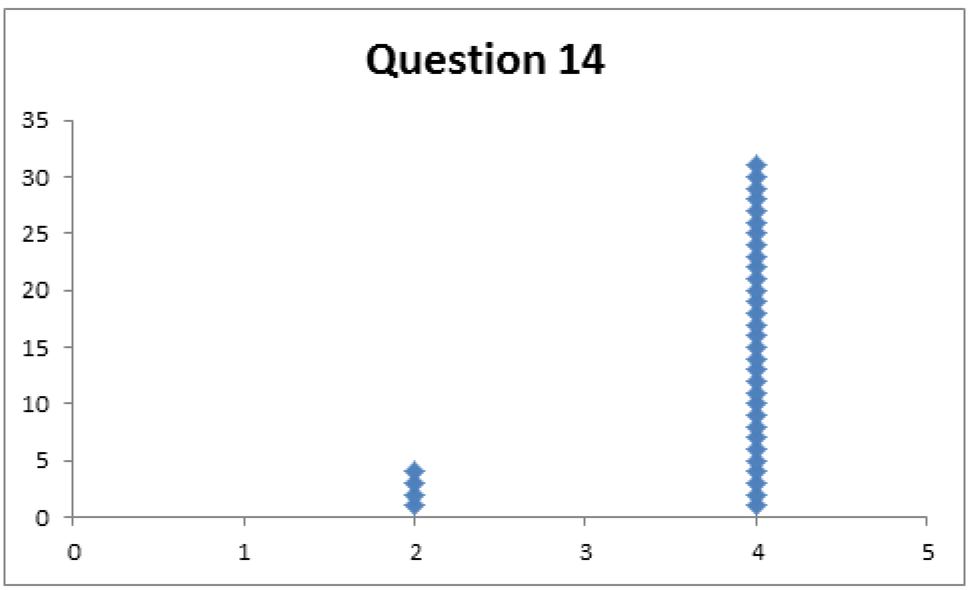


14) Sketch a graph with the incidence matrix given below. (4 points)

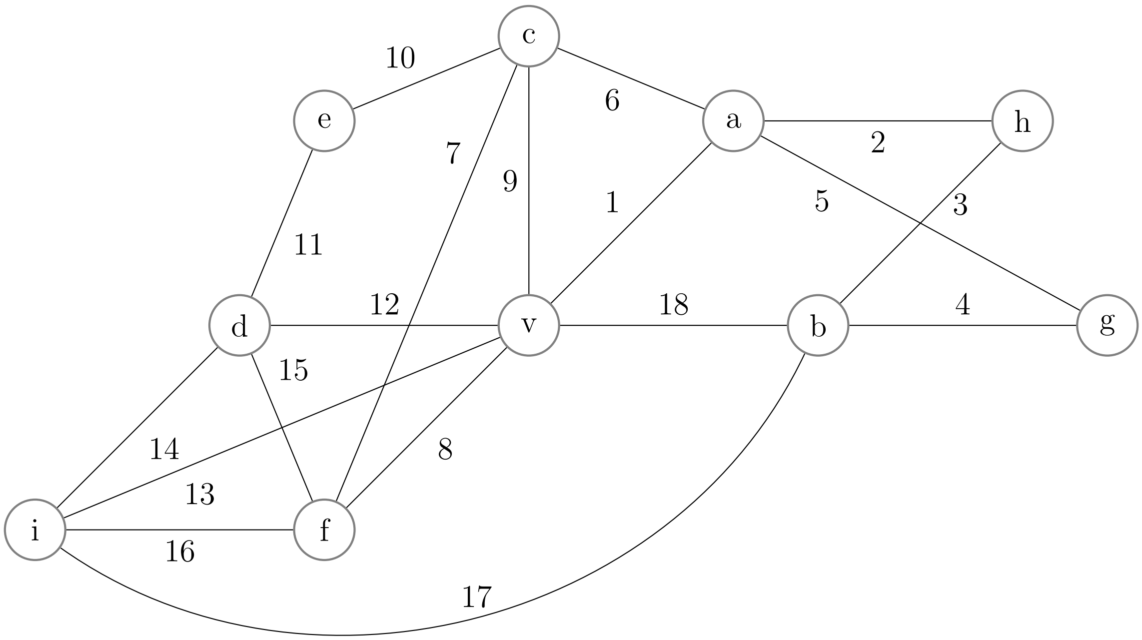
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here the rows will correspond to the vertices a, b, c, d, e and f, while the columns will correspond to the six edges.

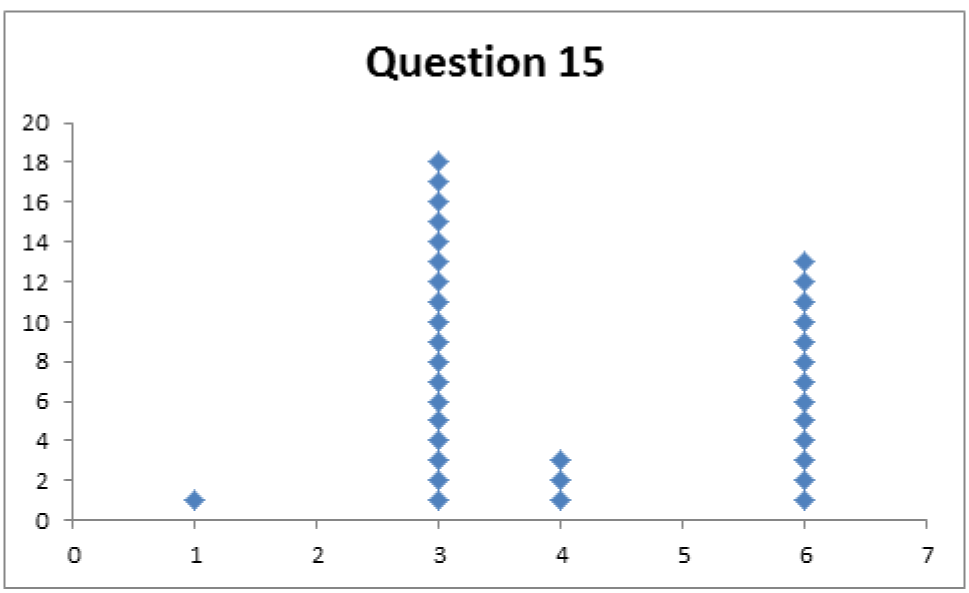




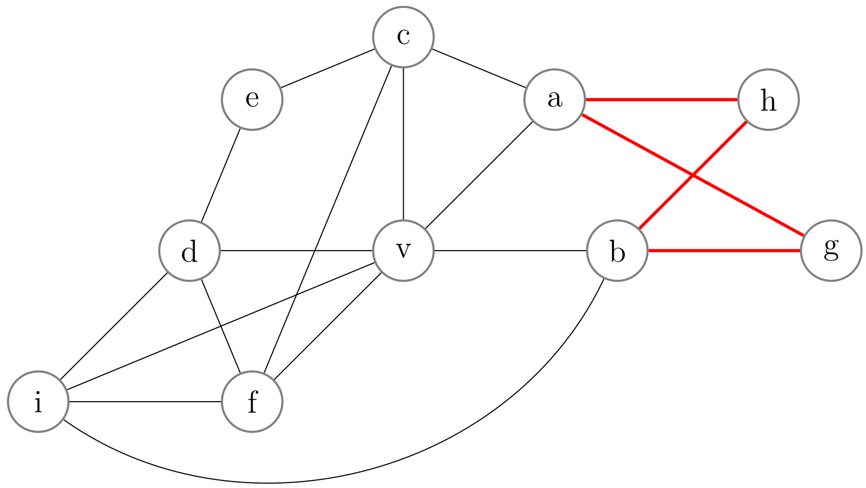
15) Using the graph below, either find and label an Eulerian tour or give a reason for why one does not exist. (6 points)



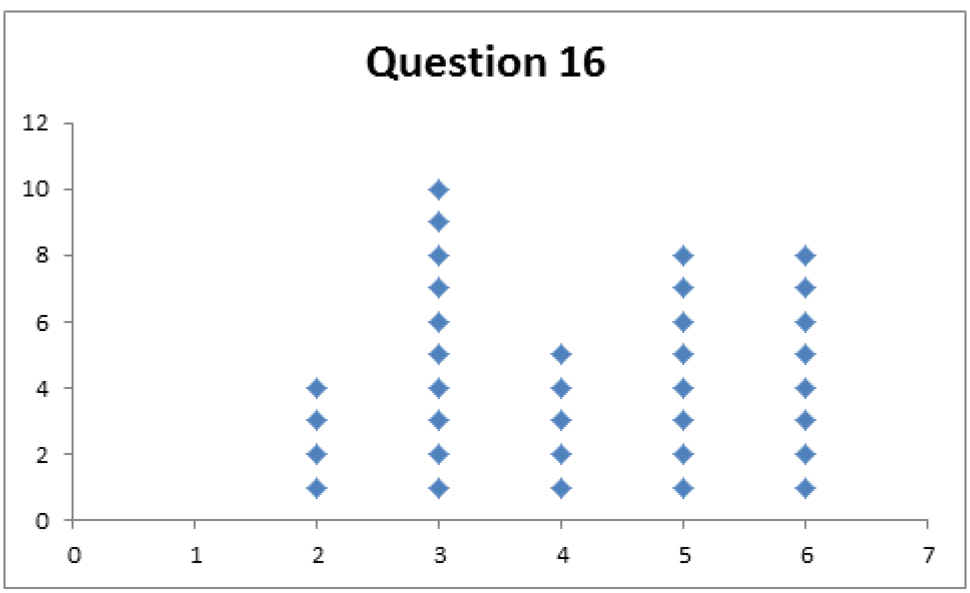
Here we have labeled the graph with a route that transverses every edge in an Eulerian tour.



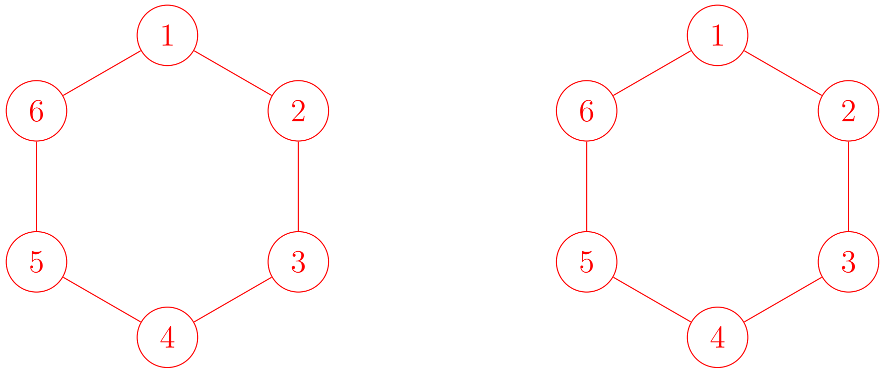
16) Using the graph below, either find and label a Hamiltonian cycle or give a reason for why one does not exist. (6 points)

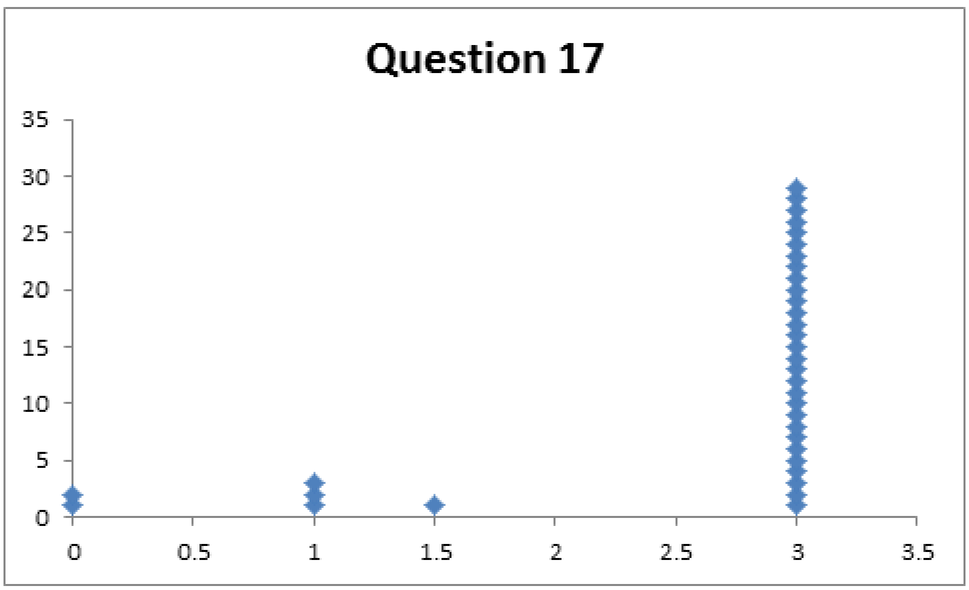


In a Hamiltonian cycle every vertex must be part of the cycle. Hence when a vertex has degree 2, both edges are part of the cycle. Here we have two vertices (h and g) that have degree 2. Hence these 4 edges must be part of the cycle. But that's already a cycle, so it cannot be made into a Hamiltonian cycle.



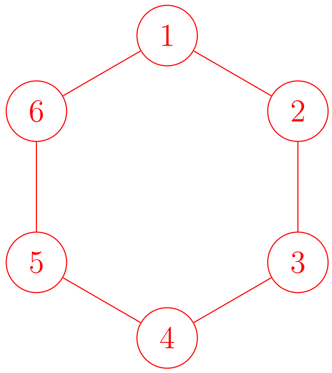
17) Sketch two graphs, each with 6 vertices, that are isomorphic. (3 points)
Here you can make any two graphs that are the same - in fact you need not even make them look different:

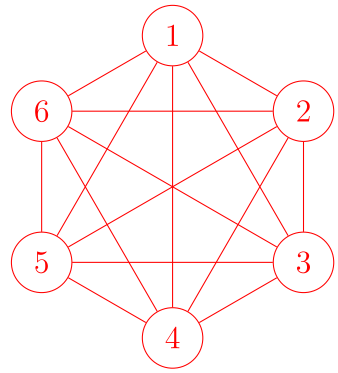


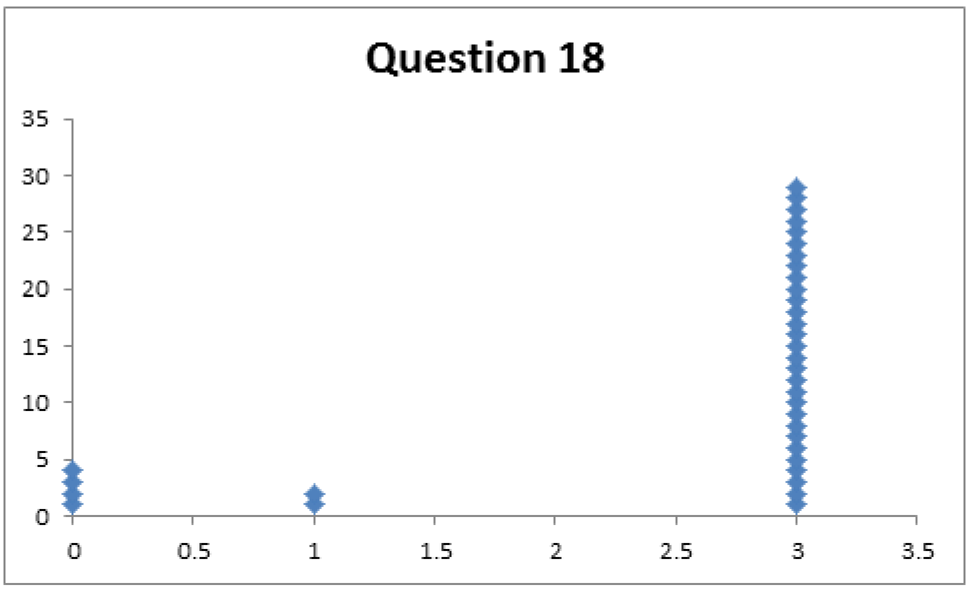


18) Sketch two graphs, each with 6 vertices, that are not isomorphic. (3 points)

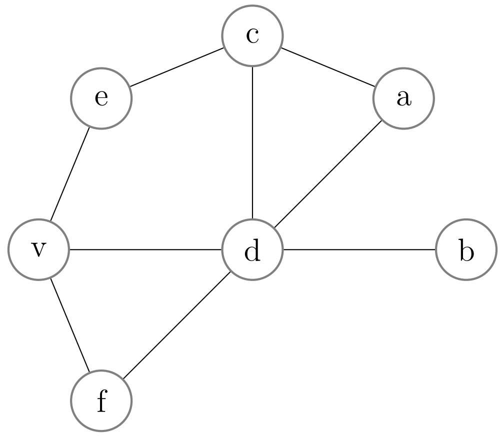
Here you can make any two graphs that are not the same, such as:



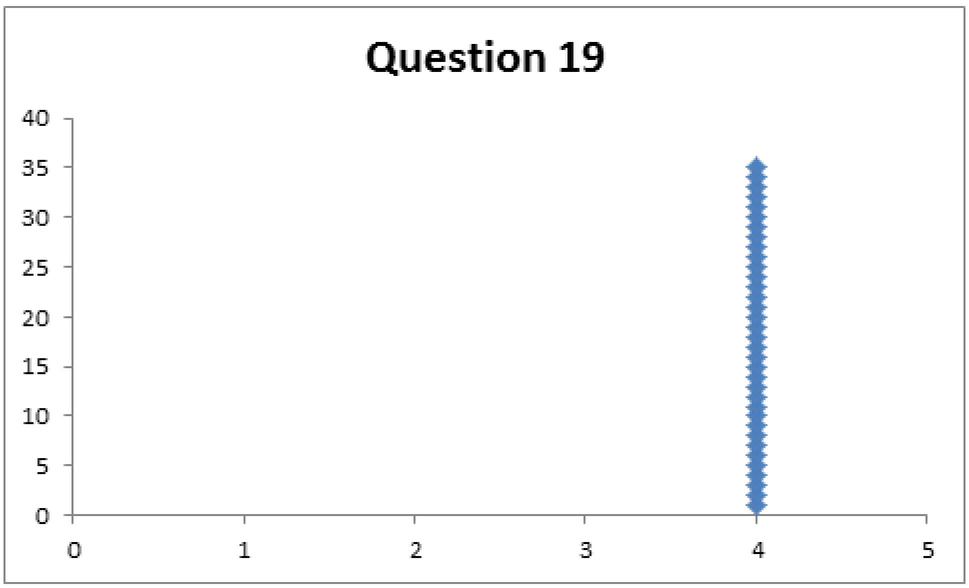




19) In the graph below, what is the degree of v ? (4 points)



Here we count up the number of vertices adjacent to v. We see that there are 3 of them



20) **Bonus Question.** Let G be a graph with n vertices and m edges. Using proper notation, give an obvious lower bound on the runtime needed to find an Eulerian tour.(2 points)

An Eulerian tour must go through every edge, so the most obvious lower bound is $\Omega(m)$.