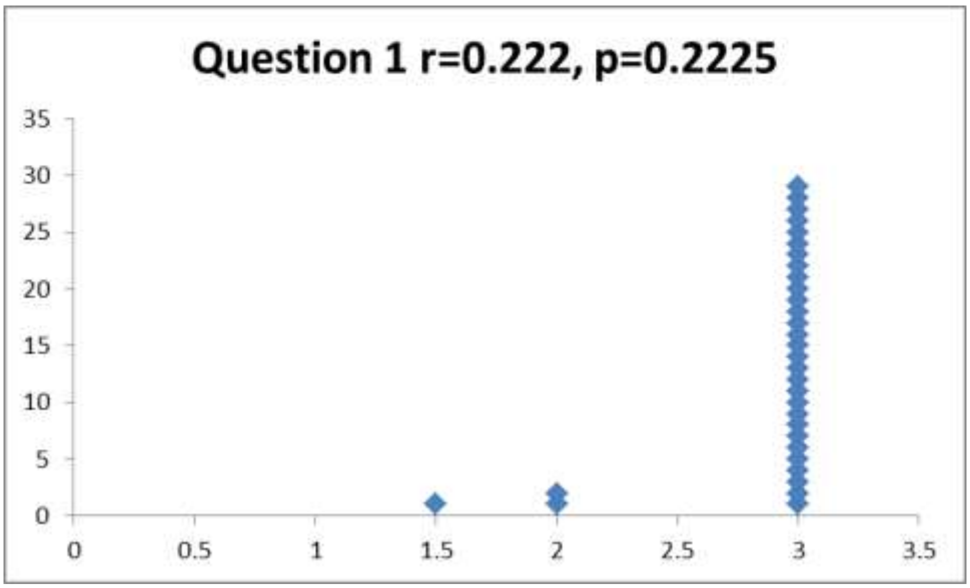
Name ______Solutions____ Discrete II, Test 1, Spring 2017

Please show all your work and circle your answer when appropriate. You do not need to simplify answers unless the problem specifies to do so.

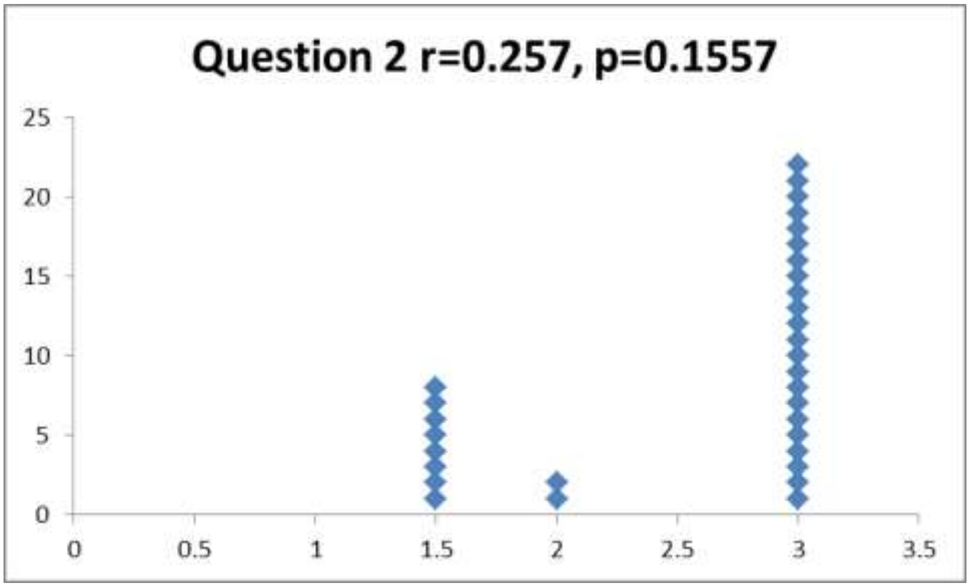
1) Given $f(n) = 2n^3 + 3n^2 + n\log(n)$, find a big-Theta ($\Theta$) notation that gives an asymptotic approximation for $f(n)$. (3 points)

$$\Theta(n^3)$$



2) Given $f(n) = 2n^3 + 3n^2 + n\log(n)$, find a little-oh (o) notation that gives an asymptotic upper bound for $f(n)$. (3 points)

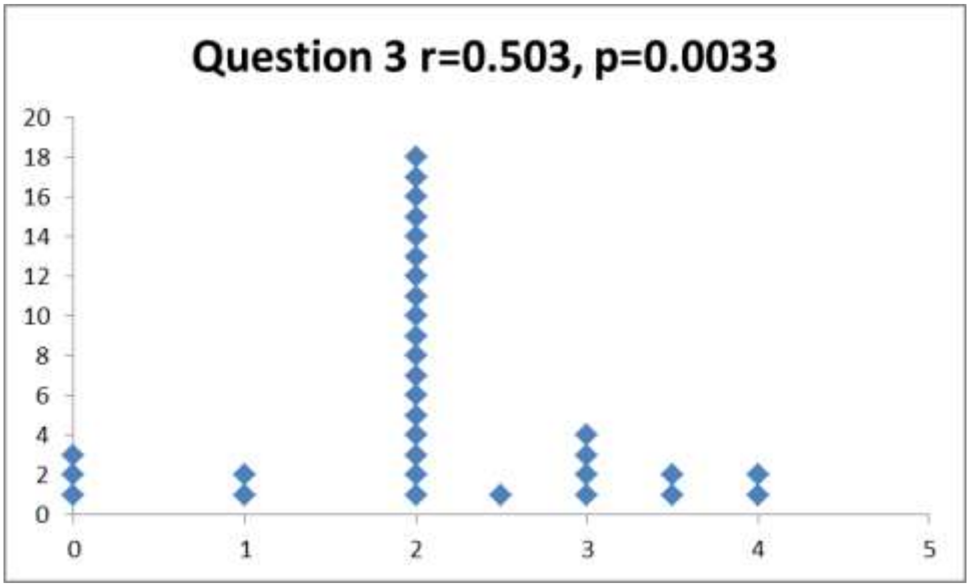
$o(n^4)$ or anything that outgrows $\Theta(n^3)$.



3) Justify the claim that $5n^3 - n^2 + 2n$ is $O(n^3)$. (4 points)

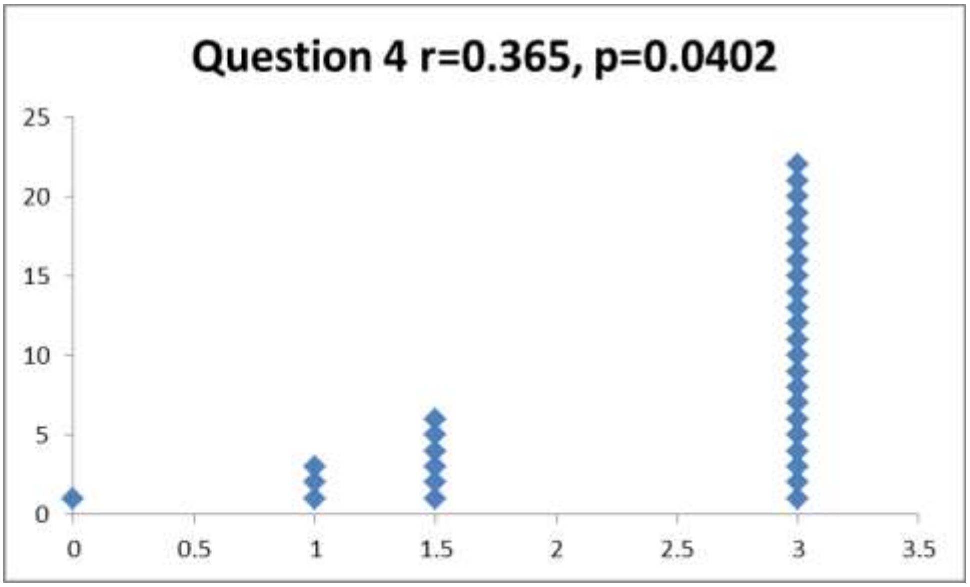
$$5n^3 - n^2 + 2n \leq 5n^3 + 2n \leq 5n^3 + n \cdot n \leq 5n^3 + n^3 = 6n^3 \text{ which is } O(n^3)$$

Note that the inequalities above require $n \geq 2$. Hence we can say that beyond $n = 2$, we know that $f(x) \leq 6n^3$ which is what it means to be $O(n^3)$.



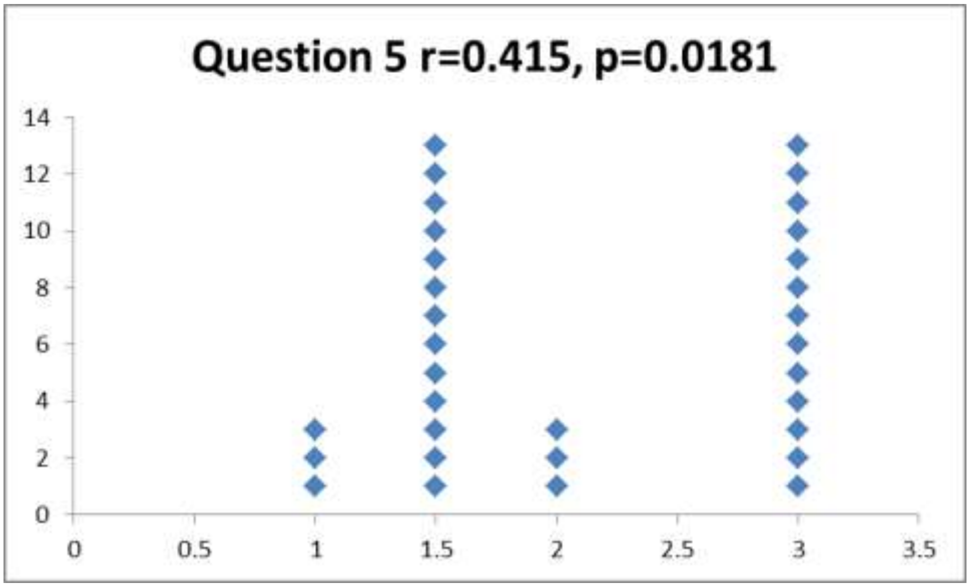
4) How many 6-character license plates can be formed from the letters and digits $\{A, B, \ldots, Z, 0, 1, \ldots, 9\}$?
(3 points)

36^6



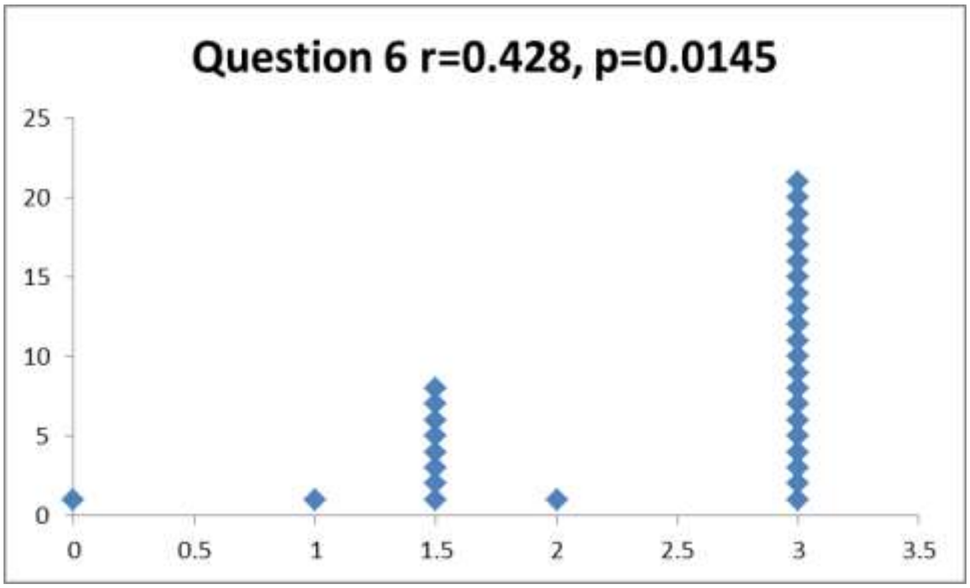
5) How many 6-character license plates can be formed from the letters and digits $\{A, B, \ldots, Z, 0, 1, \ldots, 9\}$. This time the license plate must be 4 letters followed by 2 digits. (3 points)

$$26^4 \cdot 10^2$$



6) Suppose a bag of 7 red balls, 4 blue balls, 10 green balls, and 3 tan balls. The balls will be arranged in a line on a table. How many ways can this arrangement be formed? (3 points)

$$\binom{24}{7 \quad 4 \quad 10 \quad 3}$$



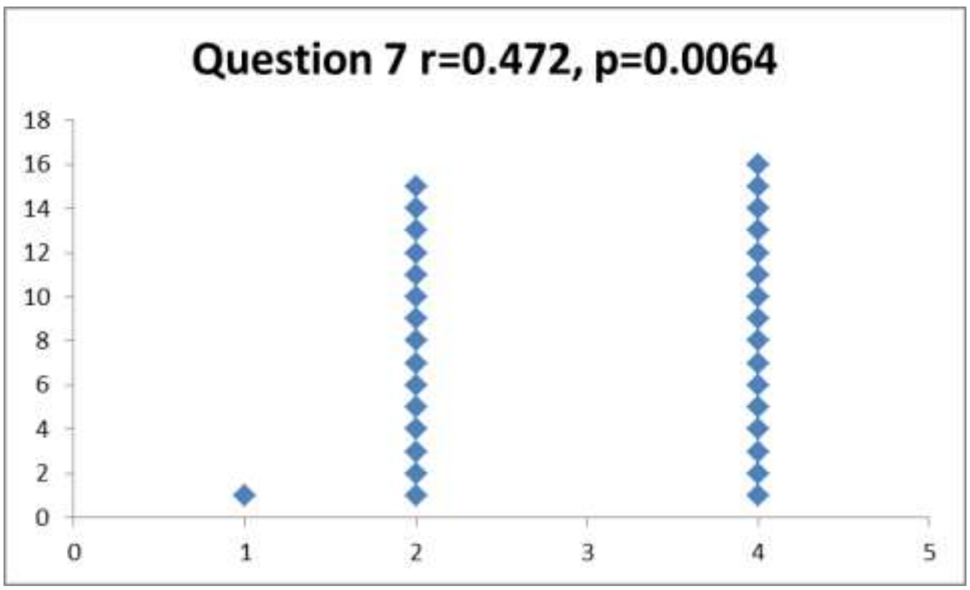
7) Consider the equation below with variables x_1, x_2, x_3, and x_4. Each variable must be an integer. x_1 and x_2 must be at least 0. However, x_3 must be at least 2 and x_4 at least 7. How many solutions does it have? (4 points)

$$x_1 + x_2 + x_3 + x_4 = 30$$

$$x_3' = x_3 - 2$$
$$x_4 = x_4' - 7$$

$$x_1 + x_2 + x_3' + x_4' = 21$$

$$\binom{21+3}{21} = \binom{24}{21} = \binom{24}{3}$$

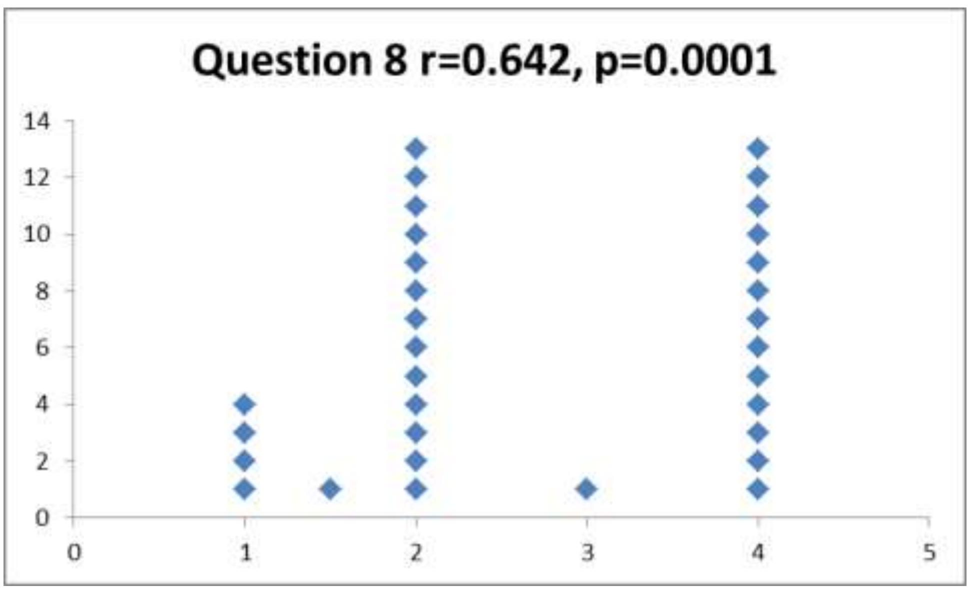


8) Consider the equation below with integer variables x_1, x_2, x_3, x_4, and x_5 subject to the accompanying constraints. How many solutions does it have? (4 points)

$$x_1 + x_2 + x_3 + x_4 + x_5 = 48$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$x_3 \geq 0$$
$$x_4 \geq 0$$
$$x_5 \geq 0$$
$$x_5 \leq 6$$

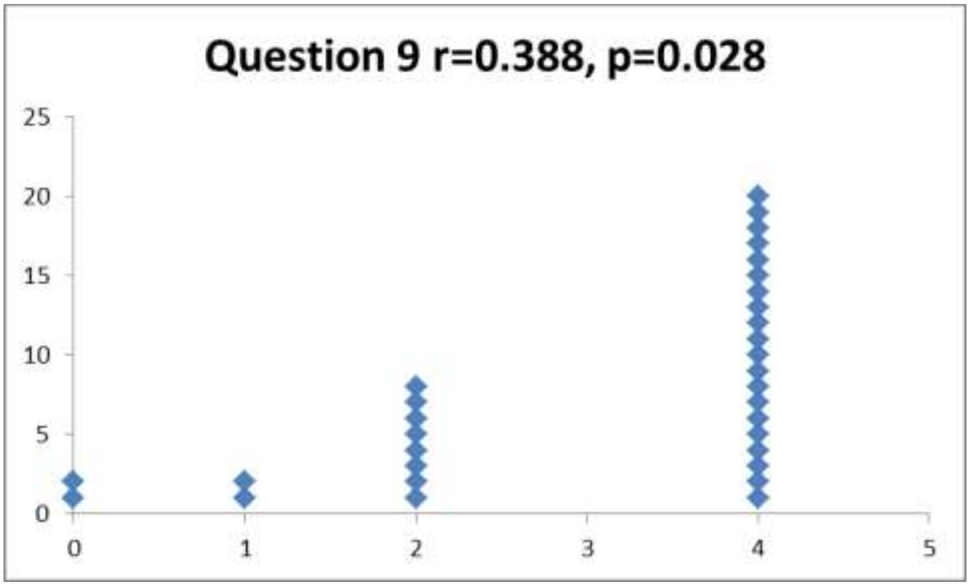
Total solutions: $\binom{48+4}{48} = \binom{52}{48} = \binom{52}{4}$
Solutions with $x_5 \geq 7$: $\binom{41+4}{41} = \binom{45}{41} = \binom{45}{4}$

Answer: $\binom{52}{4} - \binom{45}{4}$



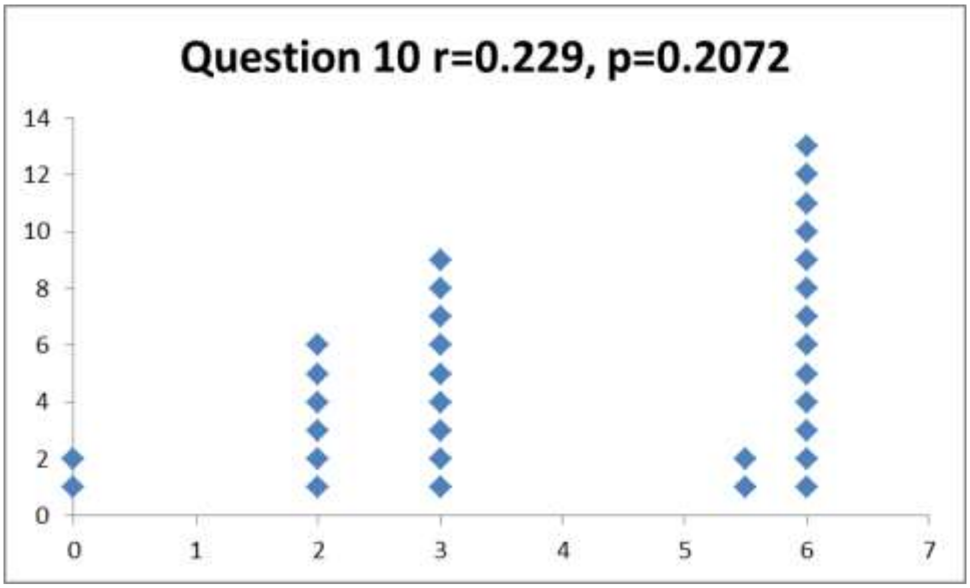
9) A bit string is a sequence of characters consisting of 0s and 1s. How many eight-bit strings consist of two 1s and six 0s? (4 points)

$$\binom{8}{2}1^2 1^6 = \binom{8}{2}$$



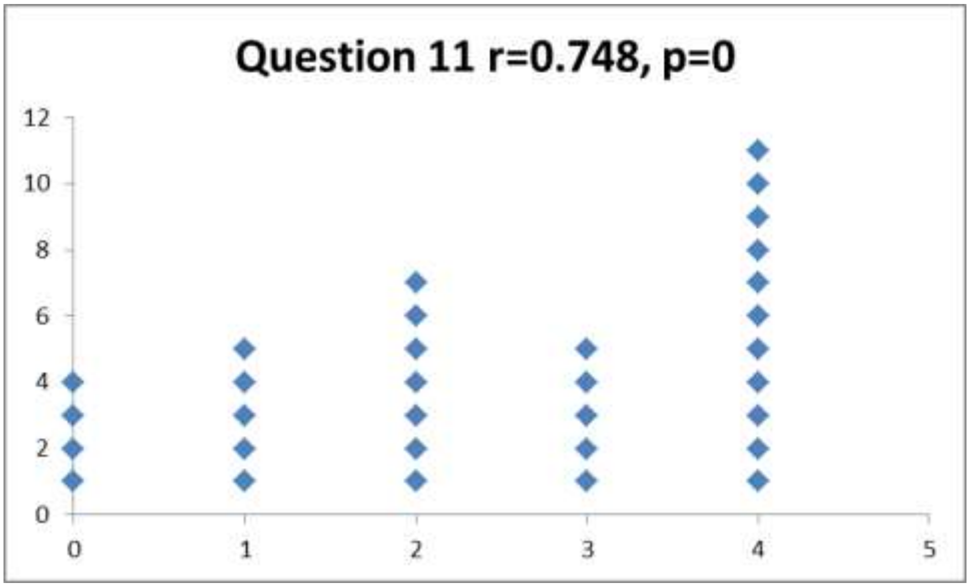
10) Calculate $\binom{9}{2}$. Do all the arithmetic until you get a single number as your answer. (6 points)

$$\binom{9}{2} = \frac{9!}{7!\,2!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 2\cdot 1} = \frac{9\cdot 8}{2\cdot 1} = 9\cdot 4 = 36$$



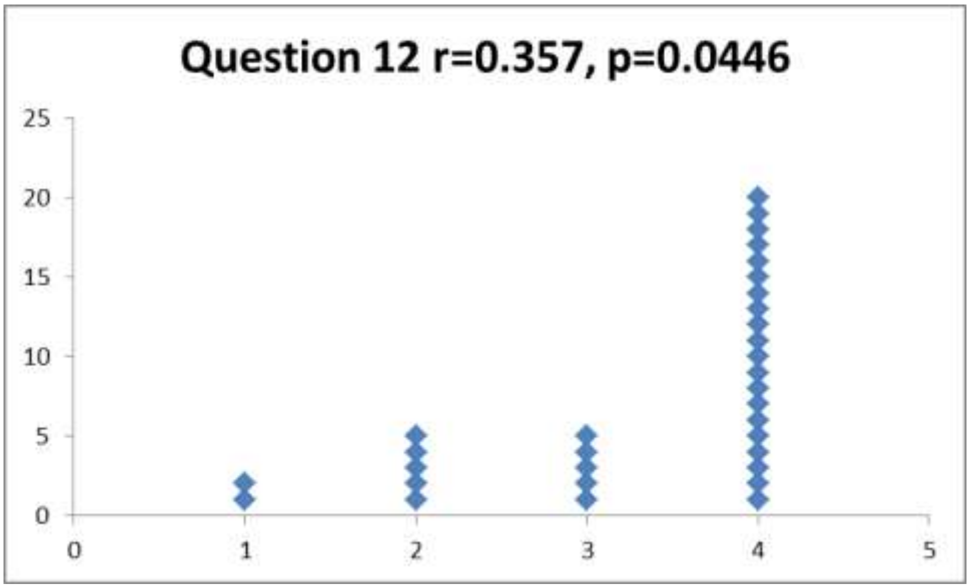
11) Find the coefficient of x^{30} in the expression $(x+5)^{120}$. (4 points)

$$\binom{120}{30}5^{90}$$



12) You are dealt 5 cards from a standard deck of 52 cards. What is the probability you get a four-of-a-kind? (A four-of-a-kind is when you have 4 of one type, and one of another, such as QQQQJ.) (4 points)

$$\frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{13 \cdot 12 \cdot 4}{\binom{52}{5}}$$

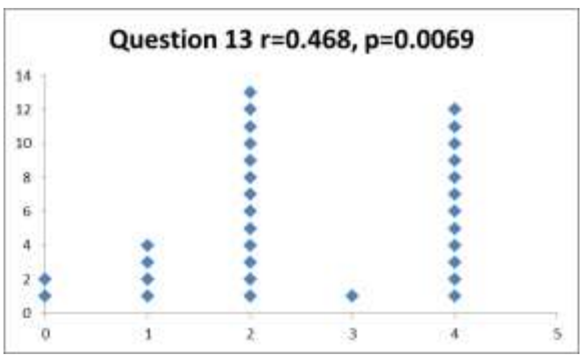


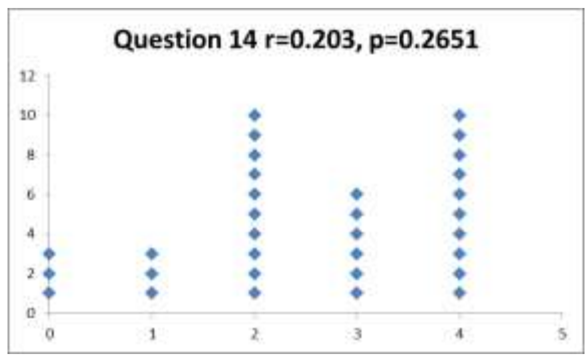
13) List the first six 4-combinations of $\{A, B, C, D, E, F\}$ in lexicographic ordering. (4 points)

A B C D
A B C E
A B C F
A B D E
A B D F
A B E F

14) List the first six 4-permutations of $\{A, B, C, D, E, F\}$ in lexicographic ordering. (4 points)

A B C D
A B C E
A B C F
A B D C
A B D E
A B D F



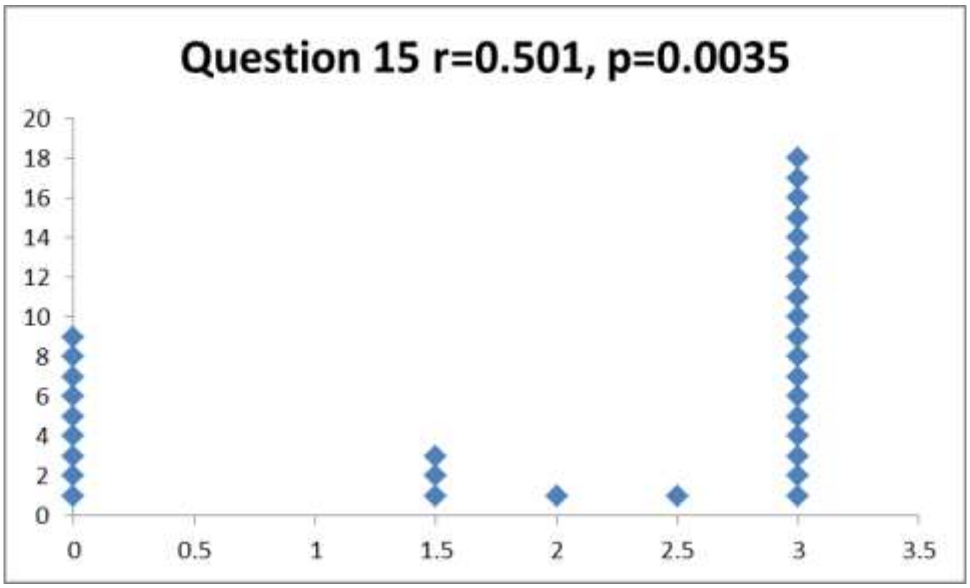


Consider the recurrence relation below for then next **four** questions.

$$a_n = 7a_{n-1} - 10a_{n-2}$$
$$a_0 = 5$$
$$a_1 = 16$$

15) What is a_2? (3 points)

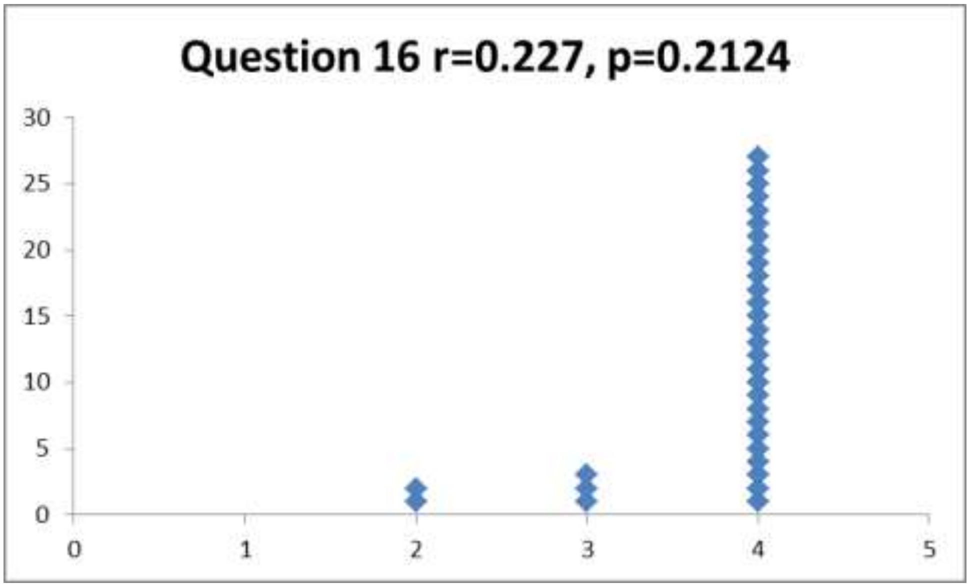
$$a_2 = 7 \cdot 16 - 10 \cdot 5 \quad (= 112 - 50 = 62)$$



16) Find the general solution to the recurrence relation. (4 points)

$$x^2 = 7x - 10$$
$$x^2 - 7x + 10 = 0$$
$$(x-5)(x-2) = 0$$
$$x = 2, 5$$

$$a_n = b \cdot 2^n + c \cdot 5^n$$



17) Find the particular solution to the recurrence relation with the given initial conditions. (4 points)

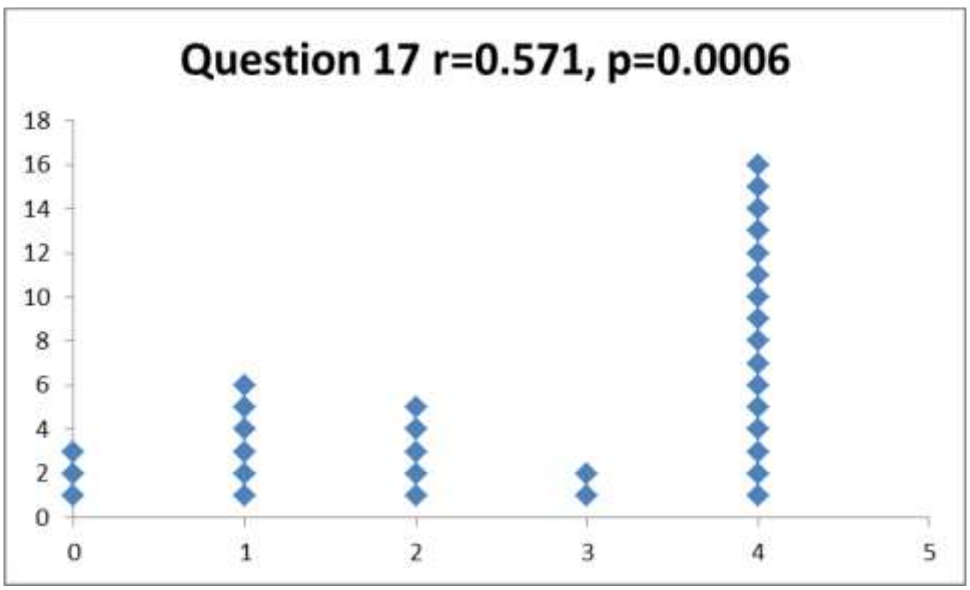
$a_0 = 5$: $5 = b + c$
$a_1 = 16$: $16 = 2b + 5c$

Using the elimination method we obtain:

$$6 = 3c$$
$$c = 2$$
$$b = 3$$

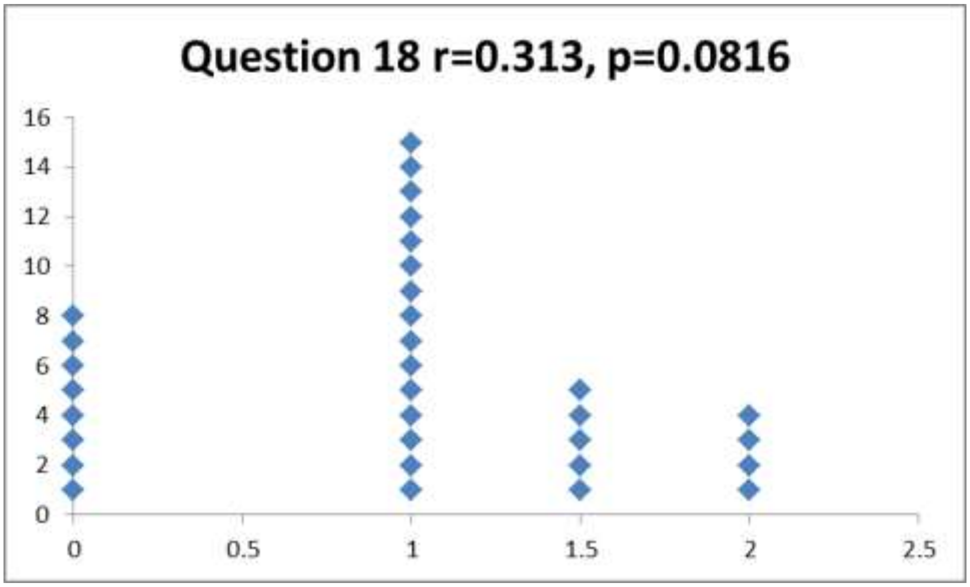
Hence the particular solution is:

$$a_n = 3 \cdot 2^n + 2 \cdot 5^n$$



18) What is the smallest upper bound on a_n, using big-Oh? (2 points)

a_n is $O(5^n)$



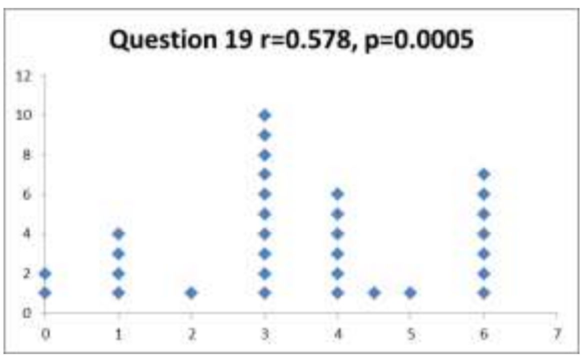
In the next **two** problems, we will represent the number of vertices by “n” and number of edges by “m”.

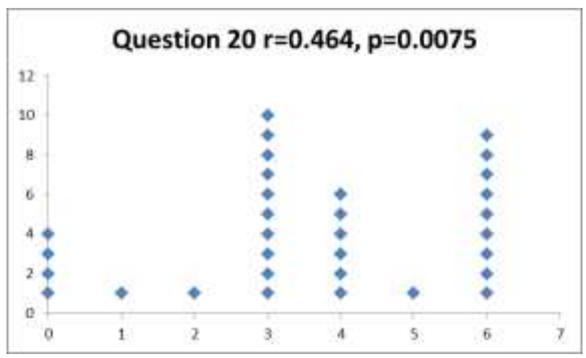
19) In Dijkstra’s algorithm on an arbitrary graph, give and explain a meaningful asymptotic upper bound for the total runtime of the algorithm. (6 points)

A decent asymptotic upper bound is $O(n^2)$. We obtain this bound implementing the algorithm by iterating through each vertex, adding it to a shortest-path-tree. At each iteration, we iterate through the remaining vertices and identify the closest one.

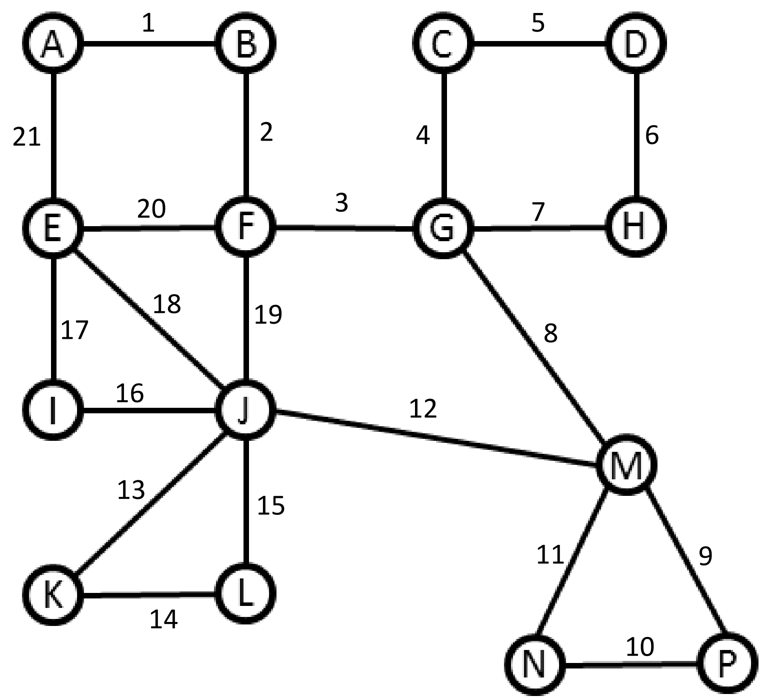
20) Give and explain a meaningful asymptotic lower bound for the total runtime required to find an Eulerian cycle in a graph. (6 points)

A lower bound for this problem is $\Omega(m)$. We obtain this by considering the fact that we must ultimately add every edge to the cycle.

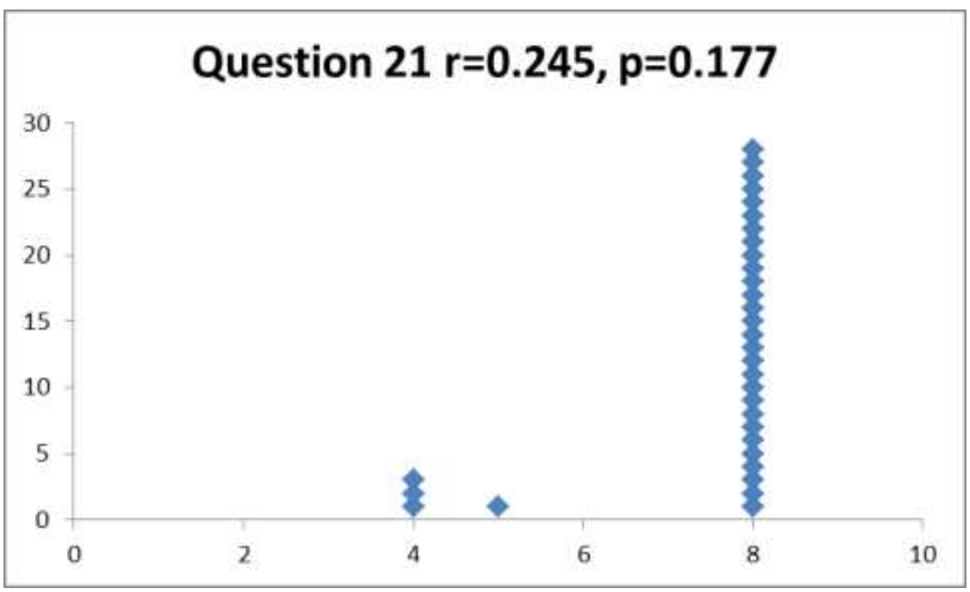




21) Using the graph below, find and label an Eulerian cycle. Label the edges on the graph in the order in which you add them to your cycle. (8 points)



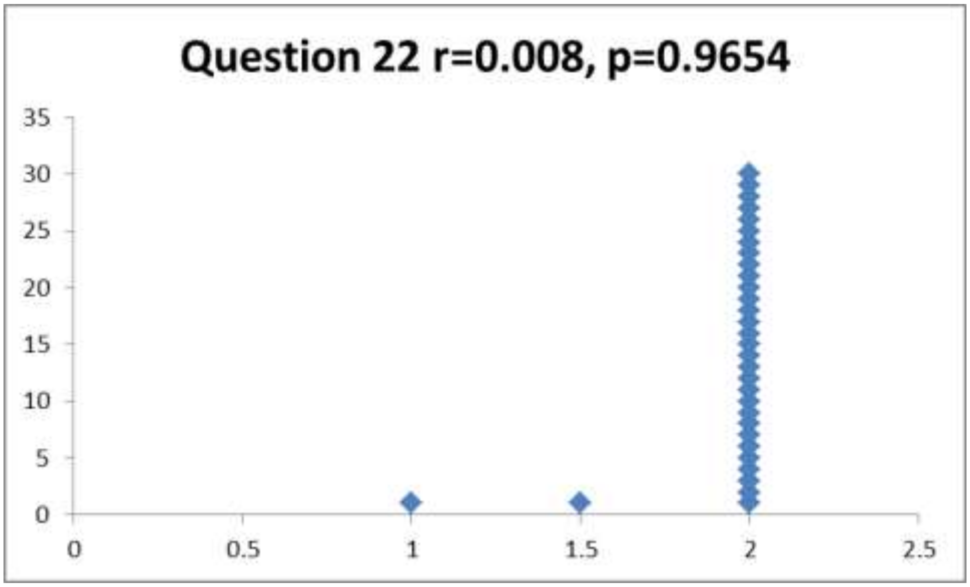
There are multiple correct answers.



22) Give an example of a simple path between A and M. (2 points)

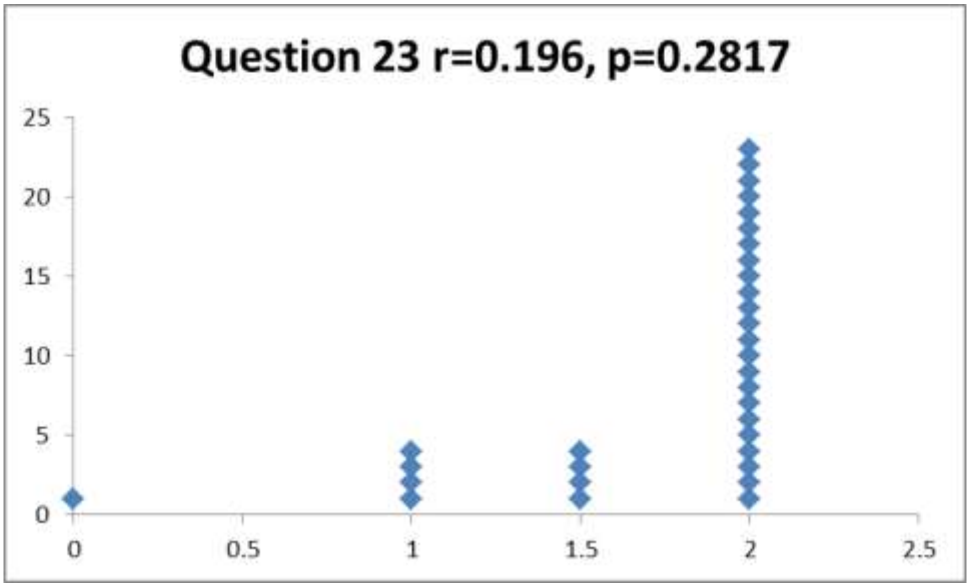
A-E-J-M.

There are multiple correct answers.



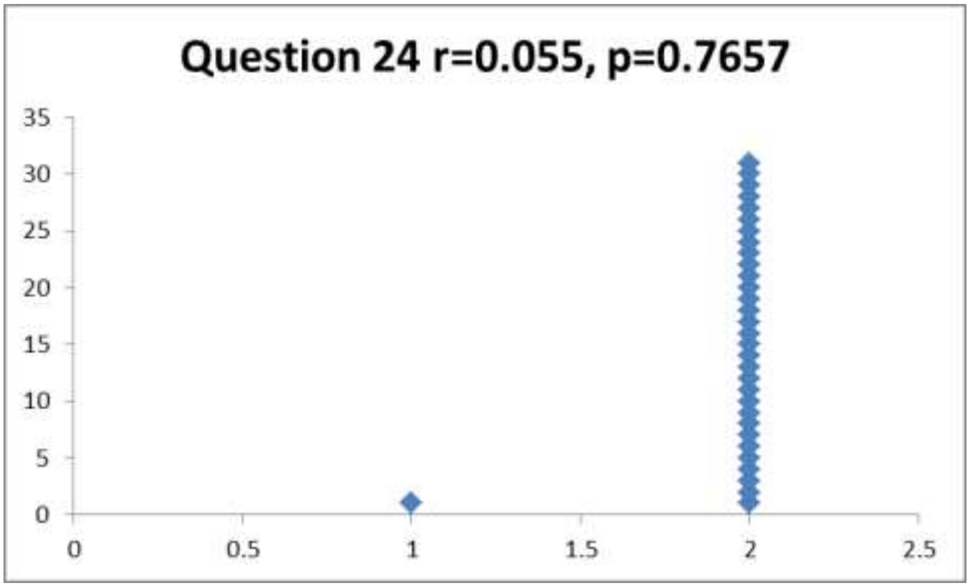
23) What is a shortest cycle through A? (2 points)

A-E-F-A or A-B-F-A



24) What vertex has the smallest degree? (2 points)

G, with degree 1.



25) Run Dijkstra's algorithm on the graph to find the shortest path between A and N. Illustrate your work on the graph itself. (8 points)