Codename ____ Fields and Rings, Test 1

(Do not put your name on the test; write your name and codename on the code sheet)

1) Consider the ideal I of $\mathbb{Q}[x]$ defined below. Write it in terms of a single generator. (100 points)

$$I = \{(x^2 + x)P + x^2Q | P, Q \in \mathbb{Q}[x]\}$$

T or F 2) Every ideal in $\mathbb{Q}[x]$ is a principal ideal. (20 points)

T or F 3) Every ideal in $\mathbb{Q}[x, y]$ is a principal ideal. (20 points)

T or F 4) Every ideal in $\mathbb{Q}$ is a principal ideal. (20 points)

5) Run through the Extended Euclidean Algorithm in $\mathbb{Z}_2[x]$ to solve the equation below: (100 points)

$$(x^3 + x^2 + 1)P + (x^2 + 1)Q = 1$$

T or F 6) $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ (20 points)

T or F 7) $\mathbb{Z}_2[x] = \mathbb{Z}/2\mathbb{Z}$ (20 points)

T or F 8) $\mathbb{Z}_2[x]/\langle x \rangle$ is a field. (20 points)

Codename ____ Sheet 2

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9) Prove or disprove the claim that $\mathbb{Q}[x]$ mod $\langle x^2 - 1\rangle$ is a field. (100 points)

10) How many elements are in $\mathbb{Q}[x]$ mod $\langle x^2 - 1\rangle$? (50 points)

11) How many elements are in $\mathbb{Z}_5[x]$ mod $\langle x^2 - 1\rangle$? (50 points)

12) Let R be a ring with unity, and S be a ring. Label as 1_R the unity in R. It is given that there is a surjective homomorphism $\varphi: R \to S$. Prove that S also has unity. (100 points)

Codename ____ Sheet 3

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13) Let ω be a special complex number such that $\omega^3 = 1$ and $\omega^3 + \omega^2 + \omega = 0$. Obviously $\mathbb{R}[\omega] \subseteq \mathbb{C}$. Prove that $\mathbb{R}[\omega]$ is closed under addition and multiplication. (100 points)

14) What is the dimension of $\mathbb{R}[\omega]$ as a $\mathbb{R}$-vector space? (30 points)

15) What is the dimension of $\mathbb{R}[\omega]$ as a $\mathbb{Q}$-vector space? (30 points)

T or F 16) $\mathbb{R}[\omega]$ is a $\mathbb{Z}_2$-vector space? (20 points)

T or F 17) $\mathbb{R}[\omega]$ is a $\mathbb{C}$-vector space? (20 points)

18) Let R be a ring with unity. It is given that $ab = bc = 1$. Prove that $a = c$. (Note that it was *not* specified that R is commutative, nor that any of a, b, or c are invertible). (100 points)

19) Let R be a ring with unity and let 1 be the identity. What does this mean? Express the definition of identity. (30 points)

20) Let R be a ring with unity and let x be an invertible element of R. What does this mean? Express the definition of invertible. (30 points)