1) Consider the ideal $I$ of $\mathbb{Q}[x]$ defined below. Write it in terms of a single generator. (100 points)

$$I = \{(x^2 + x)P + x^2 Q | P, Q \in \mathbb{Q}[x]\}$$

T or F  2) Every ideal in $\mathbb{Q}[x]$ is a principal ideal. (20 points)

T or F  3) Every ideal in $\mathbb{Q}[x, y]$ is a principal ideal. (20 points)

T or F  4) Every ideal in $\mathbb{Q}$ is a principal ideal. (20 points)
5) Run through the Extended Euclidean Algorithm in \( \mathbb{Z}_2[x] \) to solve the equation below: (100 points)
\[
(x^3 + x^2 + 1)P + (x^2 + 1)Q = 1
\]

T or F 6) \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \) (20 points)

T or F 7) \( \mathbb{Z}_2[x] = \mathbb{Z}/2\mathbb{Z} \) (20 points)

T or F 8) \( \mathbb{Z}_2[x]/\langle x \rangle \) is a field. (20 points)
9) Prove or disprove the claim that \( \mathbb{Q}[x] \mod (x^2 - 1) \) is a field. (100 points)

10) How many elements are in \( \mathbb{Q}[x] \mod (x^2 - 1) \)? (50 points)

11) How many elements are in \( \mathbb{Z}_5[x] \mod (x^2 - 1) \)? (50 points)
12) Let $R$ be a ring with unity, and $S$ be a ring. Label as $1_R$ the unity in $R$. It is given that there is a surjective homomorphism $\varphi: R \to S$. Prove that $S$ also has unity. (100 points)
13) Let $\omega$ be a special complex number such that $\omega^3 = 1$ and $\omega^3 + \omega^2 + \omega = 0$. Obviously $\mathbb{R}[\omega] \subseteq \mathbb{C}$. Prove that $\mathbb{R}[\omega]$ is closed under addition and multiplication. (100 points)

14) What is the dimension of $\mathbb{R}[\omega]$ as a $\mathbb{R}$-vector space? (30 points)

15) What is the dimension of $\mathbb{R}[\omega]$ as a $\mathbb{Q}$-vector space? (30 points)

T or F 16) $\mathbb{R}[\omega]$ is a $\mathbb{Z}_2$-vector space? (20 points)

T or F 17) $\mathbb{R}[\omega]$ is a $\mathbb{C}$-vector space? (20 points)
18) Let $R$ be a ring with unity. It is given that $ab = bc = 1$. Prove that $a = c$. (Note that it was not specified that $R$ is commutative, nor that any of $a, b, \text{ or } c$ are invertible). (100 points)

19) Let $R$ be a ring with unity and let 1 be the identity. What does this mean? Express the definition of identity. (30 points)

20) Let $R$ be a ring with unity and let $x$ be an invertible element of $R$. What does this mean? Express the definition of invertible. (30 points)