Codename ____________________________________________________ Fields and Rings, Test 2

(Do not put your name on the test; write your name and codename on the code sheet)

1) Construct an isomorphism between $\mathbb{Z}_6$ and a subring of $\mathbb{Z}_{12}$. (100 points)

(Whoops – this question turned out to be bogus. There is no subring of $\mathbb{Z}_{12}$ that is isomorphic to $\mathbb{Z}_6$, as the only subring with 6 elements is $\{0,2,4,6,8,10\}$, but that is not isomorphic to $\mathbb{Z}_6$. Note for example that $\mathbb{Z}_6$ has unity while the former does not.)

2) As an ideal of $\mathbb{Z}$, how many elements does $\langle 4 \rangle$ have? (25 points)

3) As an ideal of $\mathbb{Z}_{36}$, how many elements does $\langle 4 \rangle$ have? (25 points)

4) As an ideal of $\mathbb{Z}_{37}$, how many elements does $\langle 4 \rangle$ have? (25 points)

5) As an ideal of $\mathbb{Z}_{38}$, how many elements does $\langle 2 \rangle$ have? (25 points)

6) Denote the set of continuous functions defined on the interval $[0,5]$ as $C[0,5]$. Consider the homomorphism below. Find its kernel. (100 points) (No, that's not a typo to have both $f$ and $\varphi$; I do mean $\varphi(f) = (f(0), f(1))$.

$$\begin{aligned} \varphi: C[0,5] &\to \mathbb{R} \times \mathbb{R} \\ f &\mapsto (f(0), f(5)) \end{aligned}$$

7) Give an example of an element in $C[0,5]$. (25 points)

8) What is the dimension of $C[0,5]$ as a $\mathbb{R}$-vector space? (25 points)

9) Give an example of an element in $\mathbb{R} \times \mathbb{R}$. (25 points)

10) What is the dimension of $\mathbb{R} \times \mathbb{R}$ as a $\mathbb{R}$-vector space? (25 points)

Codename ______________________________________________________________________ Sheet 2

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11) Consider the ring $\mathbb{Z}/7\mathbb{Z}$. Prove that the coset $7\mathbb{Z} + 20$ is equal to the coset $7\mathbb{Z} + 62$. (100 points)

12) What is a much simpler ring that $\mathbb{Z}/7\mathbb{Z}$ is isomorphic to? (25 points)

13) How many elements are in $\mathbb{Z}/7\mathbb{Z}$? (25 points)

14) How many elements are in the coset $7\mathbb{Z} + 3$? (25 points)

15) Let $R$ be a commutative ring with ideals $A$ and $B$. Further assume that $B \subseteq A$. Interestingly enough $A/B \subseteq R/B$, but even more than that $A/B$ is an ideal of $R/B$, hence it makes sense to talk about the ring ${}^{R/B}\!/_{A/B}$. But what is this ring - show that ${}^{R/B}\!/_{A/B} \cong R/A$. (100 points)

(Hint: The function $f: R/B \rightarrow R/A$ given by $f(B+x) = A + x$ is obviously a homomorphism. You don't need to prove this.)

16) Write down a generic element of $R/B$. (25 points)

17) Write down a generic element of ${}^{R/B}\!/_{A/B}$. (25 points)

Codename ______________________________________________________________________ Sheet 3

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18) Find one solution to the system of congruencies below. Show your work. (100 points)

$$x \equiv 2 \mod 3$$
$$x \equiv 4 \mod 7$$
$$x \equiv 6 \mod 11$$

19) How many solutions does this system have? (25 points)

20) To what extent could you say that your solution is unique? (25 points)

21) Order the following terms in increasing order of strength. For instance, being a field is stronger than being any ol' ring, so ring should come before field. (100 points)

- Euclidean Domain (There exists a Euclidean Algorithm)
- Field
- Integral Domain
- Principal Ideal Domain (All ideals are principal)
- Ring
- Unique Factorization Domain (Every element can be factored uniquely in terms of irreducibles)