Codename ____ Fields and Rings, Test 2, Fall 2016

(Do not put your name on the test; write your name and codename on the code sheet)

1) Let R be a ring. Describe, in an English sentence, the set $\{x \in R | \forall_{r \in R}(xr = rx)$ (50 points)

Possible answer 1: The set of all objects x in R such that $xr = rx$ for every object r in R.

Possible answer 2: The set of all objects x in R that commutes with everything in R.

Possible answer 3: The set of everything in R that commutes with everything in R.

Possible answer 4: The center of R.

Problem difficulty: Medium - we discussed this in class when we were given the English (#3 above) and we constructed the set notation given in the problem.

2) Why are $x + i$ and $(1 + i)x + (-1 + i)$ associates in $\mathbb{C}[x]$? (50 points)

Note that $(1 + i)(x + i) = (1 + i)x + (1 + i)i = (1 + i)x + (-1 + i)$. Also note that because $1 + i$ is in $\mathbb{C}$, and nonzero, it is a unit. Hence $x + i$ times a unit is $(1 + i)x + (-1 + i)$. That is precisely what it means to be associates.

Problem difficulty: Easy – this was a graded homework problem.

3) Let I and J be ideals of a ring commutative ring R. We can define an operation on these two ideals we'll call "$+$" via:

$$I + J \coloneqq \{a + b | a \in I, b \in J\}$$

Prove that $I + J$ is an ideal of R. (100 points)

First note that $I + J \subseteq R$, and so we can use the criterion that it is an ideal if we can show all of the following:

1. $I + J \neq \emptyset$
2. If $x_1, x_2 \in I + J$, then $x_1 - x_2 \in I + J$.
3. If $x \in I + J$ and $r \in R$, then $xr \in I + J$.

#1: $0 \in I + J$ because $0 = 0 + 0$ and $0 \in I, 0 \in J$.

#2: Suppose $x_1, x_2 \in I + J$. Then we can write $x_1 = a_1 + b_1$ and $x_2 = a_2 + b_2$ where $a_1, a_2 \in I$ and $b_1, b_2 \in J$. Because J is a ring, we also know that $a_1 - a_2 \in I$ and $b_1 - b_2 \in J$ We then get:

$$x_1 - x_2 = a_1 + b_1 - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2) \in I + J$$

#3 Suppose $x \in I + J$ and $r \in R$. Then we can write $x = a + b$ where $a \in I$ and $b \in J$. Because I and J are ideals, we know that $ar \in I$ and $br \in J$. We then get:

$$xr = (a + b)r = ar + br \in I + J$$

Problem difficulty: Medium – this problem presented a new piece of information that you had to figure out how to deal with. However, it only required understanding that new information and working through the definition of an ideal.

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4) Let I and J be ideals of a ring commutative ring R. Refer to the previous problem for the definition of $I + J$. Prove that $I \subseteq I + J$. (50 points)

Let $x \in I$. Note that $0 \in J$, and so $x = x + 0 \in I + J$.

Problem difficulty: Hard – this problem used a new piece of information that you had to figure out how to deal with. Furthermore, it required constructing a proof that doesn't just follow from writing out the definitions.

5) Let $S \subseteq \mathbb{Z}$ be the set of all prime numbers. Is S a ring? Justify your answer. (50 points)

This is not a ring for many reasons. One reason is that it doesn't contain the additive identity 0. Another is that it's not closed under addition because $3 + 5 = 8$. Yet another is that it's not closed under multiplication because $3 \cdot 5 = 15$.

Problem difficulty: Medium – This problem required constructing a new proof that you haven't seen before. However, the objects in question are very familiar to us.

6) Let R_1 be a ring with operations "+" and "$\cdot$". Also let R_2 be a ring with operations "$\oplus$" and "$\odot$".

a) Define $R_1 \times R_2$ as a set. (5 points)

$$R_1 \times R_2 = \{(a,b) | a \in R_1, b \in R_2\}$$

Problem difficulty: Trivial – this is a fundamentals question from Transitions and uses nothing new from this class.

b) Define the standard addition operation on $R_1 \times R_2$. (5 points)

$$(a_1, b_1) + (a_2, b_2) \coloneqq (a_1 + a_2, b_1 \oplus b_2)$$

Problem difficulty: Easy – we've done this many times.

c) Define the standard multiplication operation on $R_1 \times R_2$. (5 points)

$$(a_1, b_1) \cdot (a_2, b_2) \coloneqq (a_1 \cdot a_2, b_1 \odot b_2)$$

Problem difficulty: Easy – we've done this many times.

d) Prove that $R_1 \times R_2$ has unique additive inverses under your addition operation. (35 points).

Possible answer 1: We know that $R_1 \times R_2$ is a ring under these operations, and rings have unique additive inverses.

Possible answer 2:
Note that the identity is $(0_{R_1}, 0_{R_2})$ where 0_{R_1} and 0_{R_2} are the additive identities in R_1 and R_2 respectively.

Suppose (x,y) is an arbitrary element in $R_1 \times R_2$. Then $(x,y) + (-x,-y) = (x-x, y-y) = (0_{R_1}, 0_{R_2})$. Furthermore, because $-x$ is unique and $-y$ are unique, so $(-x,-y)$ is unique as well.

Problem difficulty: Easy – especially if you remember that $R_1 \times R_2$ is a ring. Everything we've done in class is fair game to use on a test, so #1 is actually 100% correct. However, if you were unsure whether or not you can use that fact, #2 was a direct application of the fact that R_1 and R_2 are rings.

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7) In a commutative ring with unity suppose that n is the least positive integer for which we get 0 when we add 1 to itself n times; we then say R has characteristic n. If there exists no such n, we say that R has characteristic 0. For example, the characteristic of $\mathbb{Z}_5$ is 5 because $1+1+1+1+1=0$, whereas $1+1+1+1 \neq 0$.

Suppose R and S are commutative rings with unity, and there is an isomorphism $\varphi: R \to S$. Prove that if R has characteristic n, then S also has characteristic n. (100 points)

Let 1_R and 1_S be the multiplicative identities in R and S respectively. Similarly denote 0_R and 0_S as their obvious meanings. Note that $\varphi(1_R) = 1_S$ and $\varphi(0_R) = 0_S$. Then we get:

$$\underbrace{1_S + 1_S + \cdots + 1_S}_{n \text{ times}} = \underbrace{\varphi(1_R) + \varphi(1_R) + \cdots + \varphi(1_R)}_{n \text{ times}} = \varphi\left(\underbrace{1_R + 1_R + \cdots + 1_R}_{n \text{ times}}\right) = \varphi(0_R) = 0_S$$

Hence indeed S has characteristic n.

Another more difficult way to write this is below. It's more difficult because it requires truly understanding the difference between ring multiplication and scalar multiplication as repeated addition:

$$n \cdot 1_S = n \cdot \varphi(1_R) = \varphi(n \cdot 1_R) = \varphi(0_R) = 0_S$$

Problem difficulty: Easy or Hard – this was intended to be the most difficult problem on this test. It requires taking new information and constructing a proof that isn't based on just definitions. It required some ingenuity. However, this problem was also taken from the exercises in the textbook. If you happened to do that problem while studying, this became an easy problem because it's no longer new information.