Using only the definition and facts about rings below, prove the theorems below.

**Definition D1:** A ring is a set of elements with two binary operations, called addition and multiplication, such that:
- Addition is closed
- Addition is commutative
- Addition is associative
- There exists an additive identity. (Do NOT call it 0 unless we have the uniqueness theorem)
- There exist additive inverses (Do NOT call them \(-a\) unless we have the uniqueness theorem)
- Multiplication is closed
- Multiplication is associative
- Multiplication distributes over addition

**Definition D2:** Let \( R \) be a ring and \( S \subseteq R \). \( S \) is said to be a subring of \( R \) if \( S \) is itself a ring with the same operations as \( R \).

1) Let \( a, b, \) and \( c \) be elements of a ring \( R \). Assume \( a + b = a + c \), and prove that \( b = c \).

(This is theorem T1. You cannot use theorems T2+ on this problem)

**Proof:** Because \( a \in R \), we know that it has an additive inverse, \( d \in R \), such that \( a + d = d + a = I \), where \( I \) is an additive identity.

\[
\begin{align*}
a + b &= a + c \\
\therefore d + a + b &= d + a + c \\
\therefore I + b &= I + c \\
\therefore b &= c
\end{align*}
\]
**Theorem T2:** Let $a$ and $b$ be elements of a ring $R$. Then $a + x = b$ always has a unique solution.

**Theorem T3:** Let $R$ be a ring. If $a + 0_1 = a$ and $a + 0_2 = a$ for all elements $a \in R$, then $0_1 = 0_2$.

**Theorem T4:** For each element $a$ in a ring $R$, it's additive inverse is unique.

**Theorem T5:** Let $a$ be an element of a ring $R$ and denote the additive identity as $0$. Then $a \cdot 0 = 0 \cdot a = 0$.

**Theorem T6:** Let $R$ be a ring and let $a, b \in R$. Denote the additive inverse of each element $c \in R$ as $-c$, no matter what $c$ is. Then $a(-b) = (-a)b = -(ab)$.

**Theorem T7:** Let $R$ be a ring, and $S$ a subset of $R$. $S$ is a subring if and only if all of the following are satisfied for all elements $a, b \in S$:

1. $S \neq \emptyset$
2. $a, b \in S \Rightarrow a + b \in S$
3. $a, b \in S \Rightarrow a \cdot b \in S$
4. $a \in S \Rightarrow -a \in S$

2) Prove that $2\mathbb{Z}$ is a ring.

**Proof:** Note that $2\mathbb{Z} \subseteq \mathbb{Z}$, which is a ring. Hence we can use the subring theorem, T7, above.

1) $2 = 2 \cdot 1 \in 2\mathbb{Z}$, and so $2\mathbb{Z} \neq \emptyset$

Let $a, b \in 2\mathbb{Z}$, that means that we can write $a = 2k$ and $b = 2l$ for some $k, l \in \mathbb{Z}$.

2) $a + b = 2k + 2l = 2(k + l) \in 2\mathbb{Z}$.
3) $ab = (2k)(2l) = 4kl = 2(2kl) \in 2\mathbb{Z}$.
4) $-a = -2k = 2(-k) \in 2\mathbb{Z}$. 