Using only the definition and facts about rings below, answer the two problems below.

**Definition D1:** A ring is a set of elements with two binary operations, called addition and multiplication, such that:
- Addition is closed
- Addition is commutative
- Addition is associative
- There exists an additive identity. (Do NOT call it 0 unless we have the uniqueness theorem)
- There exist additive inverses (Do NOT call them \(-a\) unless we have the uniqueness theorem)
- Multiplication is closed
- Multiplication is associative
- Multiplication distributes over addition

**Definition D2:** Let \( R \) be a ring and \( S \subseteq R \). \( S \) is said to be a subring of \( R \) if \( S \) is itself a ring with the same operations as \( R \).

**Theorem T1:** Let \( a, b, \) and \( c \) be elements of a ring \( R \). If \( a + b = a + c \), then \( b = c \).

**Theorem T2:** Let \( a \) and \( b \) be elements of a ring \( R \). Then \( a + x = b \) always has a unique solution.

**Theorem T3:** Let \( R \) be a ring. If \( a + 0_1 = a \) and \( a + 0_2 = a \) for all elements \( a \in R \), then \( 0_1 = 0_2 \).

**Theorem T4:** For each element \( a \) in a ring \( R \), its additive inverse is unique.

**Theorem T5:** Let \( a \) be an element of a ring \( R \) and denote the additive identity as 0. Then \( a \cdot 0 = 0 \cdot a = 0 \).

**Theorem T6:** Let \( R \) be a ring and let \( a, b \in R \). Denote the additive inverse of each element \( c \in R \) as \(-c\), no matter what \( c \) is. Then \( a(-b) = (-a)b = -(ab) \).

**Theorem T7:** Let \( R \) be a ring, and \( S \) a subset of \( R \). \( S \) is a subring if and only if all of the following are satisfied for all elements \( a, b \in S \):
1. \( S \neq \emptyset \)
2. \( a, b \in S \Rightarrow a + b \in S \)
3. \( a, b \in S \Rightarrow a \cdot b \in S \)
4. \( a \in S \Rightarrow -a \in S \)

**Definition D2:** Let \( R \) be a ring. A multiplicative identity of \( R \) is an element \( s \in R \) such that \( sr = rs = r \) for all \( r \in R \).

**Problem 1)** Show that if a ring \( R \) has a multiplicative identity, then it is unique.

(AFTER you prove this theorem, it will justify the notation "1" for the multiplicative identity.)
**Definition D3:** Let $R$ and $S$ be rings. A function $\varphi: R \to S$ is called a ring homomorphism if it satisfies:

1. $\varphi(r + s) = \varphi(r) + \varphi(s)$ for all $r, s \in R$.
2. $\varphi(rs) = \varphi(r)\varphi(s)$ for all $r, s \in R$.

**Problem 2)** It is known that $\mathbb{Q}[x]$ and $\mathbb{Q}$ are both rings. Show that the function $\varphi: \mathbb{Q}[x] \to \mathbb{Q}$ defined by $\varphi(f) = f(0)$ is a homomorphism.

...but before you attempt problem 2, first calculate $\varphi(23x^7 - 15x^7 + 4x^3 - 2x^2 + x + 4)$ and check your answer with the instructor to make sure you understand what the $\varphi$ function does.

**Definition D4:** Let $R$ and $S$ be rings. A ring homomorphism $\varphi: R \to S$ is called a ring isomorphism if it is also one-to-one and onto. In this case $R$ and $S$ have an identical structure as rings.