Name ____ Quiz 5

Using only the definition and facts about rings below, answer the two problems below.

Definition D1: A ring is a set of elements with two binary operations, called addition and multiplication, such that:

- Addition is closed
- Addition is commutative
- Addition is associative
- There exists an additive identity. (Do NOT call it 0 unless we have the uniqueness theorem)
- There exist additive inverses (Do NOT call them $-a$ unless we have the uniqueness theorem)
- Multiplication is closed
- Multiplication is associative
- Multiplication distributes over addition

Definition D2: Let R be a ring and $S \subseteq R$. S is said to be a subring of R if S is itself a ring with the same operations as R.

Theorem T1: Let a, b, and c be elements of a ring R. If $a + b = a + c$, then $b = c$.

Theorem T2: Let a and b be elements of a ring R. Then $a + x = b$ always has a unique solution.

Theorem T3: Let R be a ring. If $a + 0_1 = a$ and $a + 0_2 = a$ for all elements $a \in R$, then $0_1 = 0_2$.

Theorem T4: For each element a in a ring R, it's additive inverse is unique.

Theorem T5: Let a be an element of a ring R and denote the additive identity as 0. Then $a \cdot 0 = 0 \cdot a = 0$.

Theorem T6: Let R be a ring and let $a, b \in R$. Denote the additive inverse of each element $c \in R$ as $-c$, no matter what c is. Then $a(-b) = (-a)b = -(ab)$.

Theorem T7: Let R be a ring, and S a subset of R. S is a subring if and only if all of the following are satisfied for all elements $a, b \in S$:

1. $S \neq \emptyset$
2. $a, b \in S \Rightarrow a + b \in S$
3. $a, b \in S \Rightarrow a \cdot b \in S$
4. $a \in S \Rightarrow -a \in S$

Definition D2: Let R be a ring. A multiplicative identity of R is an element $s \in R$ such that $sr = rs = r$ for all $r \in R$.

Problem 1) Show that if a ring R has a multiplicative identity, then it is unique.

(AFTER you prove this theorem, it will justify the notation "1" for the multiplicative identity.)

Definition D3: Let R and S be rings. A function $\varphi: R \to S$ is called a ring homomorphism if is satisfies:

1. $\varphi(r+s) = \varphi(r) + \varphi(s)$ for all $r, s \in R$.
2. $\varphi(rs) = \varphi(r)\varphi(s)$ for all $r, s \in R$.

Problem 2)

It is known that $\mathbb{Q}[x]$ and $\mathbb{Q}$ are both rings. Show that the function $\varphi: \mathbb{Q}[x] \to \mathbb{Q}$ defined by $\varphi(f) = f(0)$ is a homomorphism.

…but before you attempt problem 2, first calculate $\varphi(23x^{17} - 15x^7 + 4x^3 - 2x^2 + x + 4)$ and check your answer with the instructor to make sure you understand what the φ function does.

Definition D4: Let R and S be rings. A ring homomorphism $\varphi: R \to S$ is called a ring isomorphism if is also one-to-one and onto. In this case R and S have an identical structure as rings.