**Definition D1:** A ring is a set of elements with two binary operations, called addition and multiplication, such that:

- Addition is closed
- Addition is commutative
- Addition is associative
- There exists an additive identity. (Do NOT call it 0 unless we have the uniqueness theorem)
- There exist additive inverses (Do NOT call them \(-a\) unless we have the uniqueness theorem)
- Multiplication is closed
- Multiplication is associative
- Multiplication distributes over addition

**Definition D2:** Let \( R \) be a ring and \( S \subseteq R \). \( S \) is said to be a subring of \( R \) if \( S \) is itself a ring with the same operations as \( R \).

**Theorem T1:** Let \( a, b, \) and \( c \) be elements of a ring \( R \). If \( a + b = a + c \), then \( b = c \).

**Theorem T2:** Let \( a \) and \( b \) be elements of a ring \( R \). Then \( a + x = b \) always has a unique solution.

**Theorem T3:** Let \( R \) be a ring. If \( a + 0_1 = a \) and \( a + 0_2 = a \) for all elements \( a \in R \), then \( 0_1 = 0_2 \).

**Theorem T4:** For each element \( a \) in a ring \( R \), it’s additive inverse is unique.

**Theorem T5:** Let \( a \) be an element of a ring \( R \) and denote the additive identity as \( 0 \). Then \( a \cdot 0 = 0 \cdot a = 0 \).

**Theorem T6:** Let \( R \) be a ring and let \( a, b \in R \). Denote the additive inverse of each element \( c \in R \) as \(-c\), no matter what \( c \) is. Then \( a(-b) = (-a)b = -(ab) \).

**Theorem T7:** Let \( R \) be a ring, and \( S \) a subset of \( R \). \( S \) is a subring if and only if all of the following are satisfied for all elements \( a, b \in S \):

1. \( S \neq \emptyset \)
2. \( a, b \in S \Rightarrow a + b \in S \)
3. \( a, b \in S \Rightarrow a \cdot b \in S \)
4. \( a \in S \Rightarrow -a \in S \)

**Definition D2:** Let \( R \) be a ring. A multiplicative identity of \( R \) is an element \( s \in R \) such that \( sr = rs = r \) for all \( r \in R \). (Do NOT call it “1” until you justify that notation by proving that it is unique.)

**Theorem T8:** Let \( R \) be a ring. If \( R \) has a multiplicative identity, then it is unique.
Definition D3: Let \( R \) and \( S \) be rings. A function \( \varphi: R \rightarrow S \) is called a ring homomorphism if it satisfies:

1. \( \varphi(r + s) = \varphi(r) + \varphi(s) \) for all \( r, s \in R \).
2. \( \varphi(rs) = \varphi(r)\varphi(s) \) for all \( r, s \in R \).

Definition D4: Let \( R \) and \( S \) be rings. A ring homomorphism \( \varphi: R \rightarrow S \) is called a ring isomorphism if it is also one-to-one and onto. In this case \( R \) and \( S \) have an identical structure as rings.

Definition D5: Let \( R \) be a ring. An element \( b \neq 0 \) in \( R \) is called a zero divisor if there is another nonzero element \( a \in R \) such that \( ab = 0 \).

Definition D6: A ring that is commutative with unity and no zero divisors is called an integral domain.

Theorem T9: Let \( R \) be an integral domain and suppose \( a \neq 0 \). If \( ab = ac \), then \( b = c \).

Definition D7: Let \( R \) be a ring with unity and \( x \in R \). If there is some element \( y \in R \) such that \( xy = 1 \), we say that \( x \) is invertible, or a unit. The set of all units of \( R \) is denoted either \( U(R) \) or \( R^* \).

Definition D8: Let \( R \) be a commutative ring and \( a, b \in R \). We say that \( a \) and \( b \) are associates of each other if there is some \( u \in R^* \) such that \( a = ub \).

Definition D9: An integral domain in which every nonzero element is invertible is called a field.

Problem 1) Prove Theorem T8.

Problem 2) Find the complex conjugate of the number \( 3 + 2i \). Then show that the complex conjugate function \( f(a + bi) := a - bi \) is a homomorphism.