Codename ____ Fields and Rings, Test 1, Fall 2017

(Do not put your name on the test; write your name and codename on the code sheet)

1) Justify the fact that $6|18$.

(20 points)

$$6 \cdot 3 = 18$$

2) Construct the division equation for 17 divided by 5.

(20 points)

$$17 = 5 \cdot 3 + 2$$

3) An principle ideal $\langle p \rangle$ of $\mathbb{Z}[x]$ is the set of all polynomial multiples of the polynomial p. Formally, that is, $\langle p \rangle \coloneqq \{fp | f \in \mathbb{Z}[x]\}$. Describe, in English, the principle ideal $\langle x - 1 \rangle$.

(40 points)

This is the set of all polynomial multiples of $x - 1$.

OR

This is the set of all polynomials with a factor of $x - 1$.

OR

This is the set of all polynomials with a root of 1.

4) Use the Euclidean Algorithm to find $\gcd(17,5)$
(60 points)

$$17 = 5 \cdot 3 + 2$$
$$5 = 2 \cdot 2 + 1$$

$$\gcd(17,5) = 1$$

5) We know that $55 \equiv 35 \text{ mod } 10$. Write down two equivalent statements you can derive from this.
(I mean meaningful things you can say. Don't just say that $55 \equiv 35 \equiv 5$. Use theorems that tell us interesting things.)
(40 points)

$10|55 - 35$
$10k = 55 - 35$ for some $k \in \mathbb{Z}$
$55 = 10k + 35$ for some $k \in \mathbb{Z}$
$35 = 10k + 55$ for some $k \in \mathbb{Z}$

(Do not put your name on the test; write your name and codename on the code sheet)

6) State Gauss's Lemma.
(20 points)

If a polynomial f can be factored ($f = gh$ for some gh) over $\mathbb{Q}$ ($g, h \in \mathbb{Q}[x]$), then it can be factored over $\mathbb{Z}$ (Actually we can choose $g, h \in \mathbb{Z}[x]$)

7) Solve the equation $7x + 3 \equiv 5 \bmod 10$. Show your work.
(60 points)

$7x \equiv 2 \bmod 10$
$3 \cdot 7x \equiv 3 \cdot 2 \bmod 10$
$x \equiv 6 \bmod 10$

8) We proved that the definition of addition, below, is *well defined*. State, precisely, what it means for this to be well defined.

$$[a]_m + [b]_m = [a + b]_m$$

(40 points)

If $[a_1]_m = [a_2]_m$ and $[b_1]_m = [b_2]_m$, then $[a_1]_m + [b_1]_m = [a_2]_m + [b_2]_m$

(Also accepted would be that $[a_1 + b_1]_m = [a_1 + b_1]_m$)

9) Formally define the set $\mathbb{Q}$. Use mathematical notation.

(20 points)

$$\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$$

10) Formally define the set $\mathbb{Q}[x]$. Use mathematical notation.

(20 points)

$$\mathbb{Q}[x] = \left\{\sum_{k=0}^{n} a_k x^k : a_k \in \mathbb{Q}, n \in \mathbb{Z}_{\geq 0}\right\}$$

(Do not put your name on the test; write your name and codename on the code sheet)

11) Factor $6x^2 + 7x + 2$.
(40 points)

There are a lot of methods for factoring trinomials that aren't monic. I prefer the "AC method":

Multiply 6 and 2 to get 12.
Factor pairs of 12:
1 & 12 (add to 13)
2 and 6 (add to 8)
3 and 4 (add to 7!!)

$$6x^2 + 7x + 2 = \frac{(6x+3)(6x+4)}{6} = (2x+1)(3x+2)$$

12) Find the product below.

$$\left(\sum_{r=0}^{50} r^2 x^r\right)\left(\sum_{s=0}^{30} 3s\, x^s\right)$$

(20 points)

$$\left(\sum_{r=0}^{50} r^2 x^r\right)\left(\sum_{s=0}^{30} 3s\, x^s\right) = \sum_{r=0}^{50}\left(\left(r^2 \sum_{s=0}^{30} 3s\right) x^{r+s}\right) = \sum_{s=0}^{30}\left(\sum_{r=0}^{50} r^2\right) 3s x^{r+s} = \sum_{k=0}^{80}\sum_{l=0}^{k} l^2 3(k-l) x^k$$

13) A polynomial is called monic if the leading term is one. Prove that the product of two monic polynomials is always monic.
(100 points)

Let f and g be monic polynomials. Then we can write:

$$f = x^n + \sum_{k=0}^{n-1} a_k x^k$$
$$g = x^m + \sum_{k=0}^{m-1} b_k x^k$$

… for some $n, m \in \mathbb{Z}_{\geq 0}$ and appropriate coefficients a_k and b_k.

Then the product we see is also monic:

$$fg = x^{n+m} + x^n \sum_{k=0}^{n-1} a_k x^k + x^m \sum_{k=0}^{m-1} b_k x^k + \left(\sum_{k=0}^{n-1} a_k x^k\right)\left(\sum_{k=0}^{m-1} b_k x^k\right)$$

(Do not put your name on the test; write your name and codename on the code sheet)

14) A polynomial in two variables can have both the variable x and the variable y. It is said to be homogeneous if every term has the same degree. For example, $3x^2y - 2x^3 + xy^2$ is homogeneous of degree 3. However, $3x^2y - 2x^2 + y$ is not homogeneous

Let f and g both be homogeneous polynomials. Show that their product, $f \cdot g$, is also homogeneous.
(100 points)

Let f be a homogeneous polynomial of degree n. Thus we may write

$$f = \sum_{k=0}^{n} a_k x^k y^{n-k}$$

for some $a_k \in \mathbb{Q}$. Similarly for a homogeneous polynomial of degree m there are $b_l \in \mathbb{Q}$ so that

$$g = \sum_{l=0}^{m} b_l x^l y^{m-l}$$

Then the product is a homogeneous polynomial of degree $n + m$:

$$fg = \left(\sum_{k=0}^{n} a_k x^k y^{n-k}\right)\left(\sum_{l=0}^{m} b_l x^l y^{m-l}\right) = \sum_{k=0}^{n}\sum_{l=0}^{m} a_k b_l x^{l+k} y^{n+m-l-k}$$

15) Suppose $2x \equiv y \bmod 100$ has a solution. Show the following two claims.

A) $2|y$

(100 points)

$2x \equiv y \bmod 100$

$\therefore 100|2x - y$

$\therefore 100k = 2x - y$ for some $k \in \mathbb{Z}$

$\therefore y = 2x - 100k = 2(x - 50k)$

$\therefore 2|y$

B) $x \equiv \frac{y}{2} \bmod 50$.

(100 points)

From the above question we see that $y = 2(x - 50k)$.

$\therefore \frac{y}{2} = x - 50k$

$\therefore 50k = x - \frac{y}{2}$

$\therefore x \equiv \frac{y}{2} \bmod 50$