Name ____ Fields and Rings, Test 2, Fall 2017

1) Prove the following theorems:

- T1
- T3
- T8
- T9
- T12
- T14a
- T16a
- T17a

(100 points each)

2) Let R be a commutative ring and S and T ideals of R. Define $J \coloneqq \{a + b | a \in S, b \in T\}$. Prove that J is an ideal of R.
(100 points)

3) Compute $4a$ and a^2 in $\mathbb{Q}[x]$ for $a = 1 + 3x^2$.
4) Compute $4a$ and a^4 in $\mathbb{Z}_7$ for $a = 2$.
(100 points)

5) Find all the subrings of $\mathbb{Z}_{12}$.
6) Consider $\varphi: \mathbb{Z}: \to 2\mathbb{Z}$ given by $\varphi(n) = 2n$. Explain why φ is not a ring homomorphism.
(100 points)

7) Write the number $e^{\frac{i\pi}{4}}$ in rectangular coordinates as $a + bi$.
8) Show that in $\mathbb{Q} \times \mathbb{Z}$, the elements $(2, -1)$ and $(4,1)$ are associates.
(100 points)

9) Explain why a field is always a PID, practically by default.
10) Give a nice description of the ideal $\langle\sqrt{7}\rangle$ in the ring $\mathbb{Z}[\sqrt{7}]$.
(100 points)

11) Factor $x^3 - 2$ into irreducibles in $\mathbb{Q}[x]$.
12) Let $\mathbb{F}$ be a field. Could the ring $\mathbb{F}[x]$ be a field? Why or why not?
(100 points)

13) Use the ring $R = \mathbb{Z}[\sqrt{2}]$ for this problem. Simplify the ideal $\langle 3 + 8\sqrt{2}, 7\rangle$ in this ideal.
(100 points)