Name ____ Test 1, Fall 2021

Part 1: Basic Knowledge

1) Let a and b be integers. What does $a|b$ mean? Be precise. (4 points)

$b = ak$ for some $k \in \mathbb{Z}$

2) Let $f, g \in \mathbb{R}[x]$. State the division equation, also known as the remainder theorem. (4 points)

$f = gq + r$ for some $q, r \in \mathbb{R}[x]$ where $\deg(r) < \deg(g)$

Or

$g = fq + r$ for some $q, r \in \mathbb{R}[x]$ where $\deg(r) < \deg(f)$

3) Let $f = \sum_{k=0}^{n} a_k x^x$. What does the rational root theorem say? Be precise. (8 points)

If $f\left(\frac{p}{q}\right) = 0$ where $p, q \in \mathbb{Z}$, then $p|a_0$ and $q|a_n$.

Part 2: Basic Skills and Concepts

4) Answer each of the following as irreducible (I), reducible (R), or not applicable (N) in $\mathbb{Z}[x]$. (8 points)

I R N a) $f = 0$ N

I R N b) $f = 1$ N

I R N c) $f = 5$ N

I R N d) $f = x$ I

I R N e) $f = x^2$ R

I R N f) $f = x^2 - 25$ R

I R N g) $f = x^2 + 25$ I

I R N h) $f = (x^2 - 2)(x^2 - 4)$ R

I R N i) $f = (x^2 + 2)(x^2 + 4)$ R

I R N j) $f = x^{123} - 1$ R

Grading note: (b) and (c) were not marked incorrect because I felt like we didn't address this point enough during class.

5) Let $f = 3x^2 - 2x + 1$. Give each of the following: (4 points)

a) What is the degree of f?

2

b) List the coefficients, separated by commas.

$3, -2, 1$

c) List the terms, separated by commas.

$3x^2, -2x, 1$

6) To which of the polynomials below does Eisenstein's theorem apply? Y=yes, N=no (4 points)

Y N a) $x^6 + 2x^4 + 4x^2 + 6$ Y

Y N b) $x^6 + 2x^4 + 4x^2 + 8$ N

Y N c) $2x^6 + 4x^4 + 6x^2 + 10$ N

Y N d) $2x^6 + 6x^4 + 12x^2 + 24$ Y

Y N e) $x^2 + 1$ N

7) Write down an expression that gives the x^{17} term of the product below. (4 points)

$$\left(\sum_{k=0}^{40} 2^k x^k\right)\left(\sum_{l=0}^{32} 3^l x^l\right)$$

$$\sum_{j=0}^{17} 2^j 3^{17-j}\, x^{17} \text{ or } \sum_{j=0}^{17} 2^{17-j} 3^j\, x^{17}$$

8) Divide $x^3 + 1$ by $x^2 - 4$. State your answer using the division equation. (4 points)

[Work is difficult to type]

$x^3 + 1 = (x^2 - 4)(x) + (4x + 1)$

9) Find the GCD of 15 and 70 using the Euclidean Algorithm. No credit will be given if you solve it by inspection and not the EA. (4 points)

$17 = 15 \cdot 4 + 10$

$15 = 10 \cdot 1 + 5$

$10 = 5 \cdot 2 + 0$

$\gcd(15{,}70) = 5$

10) Use the Extended Euclidean Algorithm to solve $15x + 70y = 5$. No credit will be given if you solve it by inspection and not the EEA. (4 points)

$10 = 70 - 15 \cdot 4$

$5 = 15 - 10 = 15 - (70 - 15 \cdot 4) = 15 \cdot 5 - 70$

$x = 5, y = -1$

11) Obtain a new function by clearing all the denominators of the function below. (4 points)

$$f(x) = \frac{1}{2}x^3 + \frac{1}{3}x^2 + \frac{5}{6}x - \frac{1}{2}$$

$$g(x) = 6f(x) = 3x^3 + 2x^2 + 5x - 3$$

Or

$$h(x) = 72f(x) = 36x^3 + 24x^2 + 60x - 36$$

Odd that a lot of people chose to use 36= $2 \cdot 3 \cdot 6$. There's no theoretical reason why you would want to use 36. On the other hand, 6 is useful because it is the LCM and 72 is useful because it is the product. But 36? Weird and useless, but I gave full credit for it because it technically works.

Part 3: Proofs

12) Let $f, g \in \mathbb{Z}[x]$. Prove that $\deg(fg) = \deg(f) + \deg(g)$ (16 points)

Write out notation for f and g as below, where $n, m, a_k, b_l \in \mathbb{Z}$ and $n, m \geq 0$.

$$f = \sum_{k=0}^{n} a_k x^k$$

$$g = \sum_{l=0}^{n} b_l x^l$$

$$fg = \left(\sum_{k=0}^{n} a_k x^k\right)\left(\sum_{l=0}^{n} b_l x^l\right) = a_n b_m x^{n+m} + (a_{n-1} b_m + a_n b_{m-1}) x^{n+m-1} + \cdots + a_0 b_0$$

From this expansion ,we see that the degree of the right hand side, and thus the degree of fg is $n + m$.

13) Let $f \in \mathbb{Z}[x]$, and write $f = a_0 + a_1 x + \cdots + a_n x^n$. Assume If $p,$ and q are coprime integers such that $f\left(\frac{p}{q}\right) = 0$. State and prove half of the rational root theorem. (16 points)

Half of the rational root theorem is that $p|a_0$.

$$f\left(\frac{p}{q}\right) = a_0 + a_1\frac{p}{q} + \cdots + a_n\left(\frac{p}{q}\right)^n$$
$$0 = a_0 + a_1\frac{p}{q} + \cdots + a_n\left(\frac{p}{q}\right)^n$$
$$0 = a_0q^n + a_1pq^{n-1} + \cdots + a_np^n$$
$$a_0q^n = -a_1pq^{n-1} - \cdots - a_np^n$$

Clearly $p| - a_1pq^{n-1} - \cdots - a_np^n$
Thus $p|a_0q^n$
However, $p \nmid q^n$
Therefore $p|a_0$

(You could have also done the other half)

14) Let $f, g_1, g_2, g_3 \in \mathbb{Z}[x]$ and assume f is prime. Prove that if $f|g_1 g_2 g_3$, then either $f|g_1$, $f|g_2$, or $f|g_3$. (16 points)

Assume $f|g_1 g_2 g_3$.
$\therefore f|(g_1 g_2) g_3$
$\therefore f|g_1 g_2$ or $g|f_3$ because f is prime.
$\therefore f|g_1$ or $f|g_2$ or $f|g_3$ because f is prime.