Name ____ Test 2, Fall 2021

Part 1: Basic Knowledge

1) Let a an element of $\mathbb{Z}_n$. What does it means for a to be invertible? Precisely state the definition.
(5 points)

2) What is a power series? Give an expression for an arbitrary element of $\mathbb{R}[\![x]\!]$. (5 points)

3) Let $\otimes$ be a binary operation on a set S. What does it mean for S to be commutative? Precisely state the definition. (5 points)

4) Consider multiplication in $\mathbb{Z}_n$. What does it mean for multiplication to be well defined? Precisely state the definition. (5 points)

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Part 2: Basic Skills and Concepts

5) Find the product below, express your answer using the sum-of-degree method. (5 points)

$$\left(\sum_{k=0}^{\infty} 2^k x^k\right)\left(\sum_{k=0}^{\infty} 3^k x^k\right)$$

6) Find the multiplicative inverse of $\sum_{k=0}^{\infty} \frac{2^k x^k}{k!}$. (5 points)

7) Find 4^{-1} mod 11. (5 points)

8) Below is a derivation of the quadratic formula. It does not work in $\mathbb{Z}_n$. Which is the first step that is not guaranteed to work in $\mathbb{Z}_n$ and why? (5 points)

$$ax^2 + bx + c = 0$$
$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + c - \frac{b^2}{4a} = 0$$
$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 = \frac{b^2}{4a} - c$$
$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 = \frac{b^2-4ac}{4a}$$
$$\sqrt{a}x + \frac{b}{2\sqrt{a}} = \frac{\pm\sqrt{b^2-4ac}}{2\sqrt{a}}$$
$$\sqrt{a}x = -\frac{b}{2\sqrt{a}} \pm \frac{\sqrt{b^2-4ac}}{2\sqrt{a}}$$
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$
$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

9) Let A, and B be square matrices. Solve $AX + X = B$ for X. Assume the needed inverse exists. (5 points)

10) Find $2 \otimes 1$, given the binary operation below. (5 points)

$\otimes$	1	2	3
1	1	1	2
2	3	1	1
3	2	3	2

Part 3: Proofs

Do not use any advanced theorems that would create circular logic.

11) Let A and B be invertible matrices in $\mathbb{R}^{n \times n}$. Prove that $(AB)^{-1}$ exists. (15 points)

12) Let S be a set with a binary operation $\otimes$. If $a \otimes e = e \otimes a = a$ and $a \otimes f = f \otimes a = a$ for all $a \in S$, Prove that $e = f$. (15 points)

13) Let R be a ring and $a, b \in R$. Prove that $-(-a) = a$. (15 points)

14) Let p be prime and $a, b \in \mathbb{Z}_p$. Prove that if $ab \equiv 0$, then either $a \equiv 0$ or $b \equiv 0$. This is called the zero product property. (15 points)