Name ____ Test 3, Fall 2021

Part 1: Basic Knowledge

1) Let R be a ring. What does it mean for addition to be closed?
(5 points)

If $a, b \in R$, then $a + b \in R$.

2) Let R be a ring and S a subset of R. State one of the subring criterions. (5 points)

S is a ring iff all of the following are satisfied:

1. $S \neq \emptyset$
2. $a, b \in S \Rightarrow a + b \in S$ (Closed under addition)
3. $a, b \in S \Rightarrow a \cdot b \in S$ (Closed under multiplication)
4. $a \in S \Rightarrow -a \in S$ (Closed under additive inverses)

Or iff all of the following are satisfied:

1. $S \neq \emptyset$
2. $a, b \in S \Rightarrow a - b \in S$ (Closed under subtraction)
3. $a, b \in S \Rightarrow a \cdot b \in S$ (Closed under multiplication)

Or iff all of the following are satisfied:

1. $S \neq \emptyset$
2. $a, b \in S \Rightarrow a + b \in S$ (Closed under addition)
3. $a, b \in S \Rightarrow a \cdot b \in S$ (Closed under multiplication)
4. S is finite.

3) Let R be a ring with unity. What is the definition of 1_R that makes it so special? (5 points)

Let $r \in R$. $1_R r = r 1_R = r$ for all $r \in R$.

Note that we need both directions because we are *not* given that R is commutative.

4) Let R be a ring and S a subring. What more does it take for S to be an ideal? (5 points)

For all $s \in S$ and $r \in R$, both $sr \in S$ and $rs \in S$.

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Part 2: Basic Skills and Concepts

5) Consider the ring $\mathbb{Z}_{24}$. Find $\langle 4 \rangle + 3$, what is it? (5 points)

If we write $\mathbb{Z}_{24} = \{0,1,2,3,4,\cdots,23\}$, then we get:
$\langle 4 \rangle = \{0,4,8,12,16,20\}$
$\langle 4 \rangle + 3 = \{3,7,11,15,19,23\}$

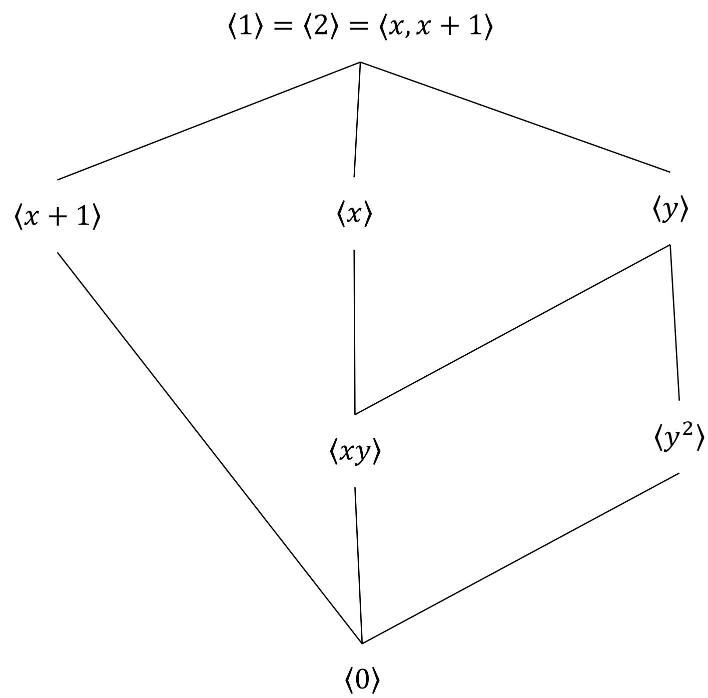
6) Let R be an integral domain with unity. True or false: $\langle 2_R \rangle$ is always a maximal ideal, and why? (5 points)

False. Consider a counterexample: $\mathbb{R}$, because $\langle 2_R \rangle = \mathbb{R}$ and so is not maximal.

Or go the other way, consider $\mathbb{Z}[x]$, in this case $\langle 2_R \rangle$ is not maximal because $\langle 2_R \rangle \subseteq \langle 2, x \rangle$.

7) Place these ideals into the same subring tree on the ring $R = \mathbb{R}[x, y]$. Note that they are not all distinct. (5 points)

a. $\langle 0 \rangle$
b. $\langle 1 \rangle$
c. $\langle 2 \rangle$
d. $\langle x \rangle$
e. $\langle x + 1 \rangle$
f. $\langle x, x + 1 \rangle$
g. $\langle y \rangle$
h. $\langle y^2 \rangle$
i. $\langle xy \rangle$



8) Let R be a ring with unity. Define $2_R \coloneqq 1_R + 1_R$. In most rings, $2_R \neq 1_R$. What is the condition required to guarantee that $2_R \neq 1_R$? (5 points)

$0_R \neq 1_R$, or equivalently that R is nontrivial.

Part 3: Proofs

Do not use any advanced theorems that would create circular logic.

9) Let R be an integral domain. State and prove what is known as the cancellation law. (20 points)

Let $a, b, c \in R$ such that $a \neq 0$. If $ab = ac$, then $b = c$.

Assume $ab = ac$.

$\therefore ab - ac = 0_R$	by subtraction.
$\therefore a(b - c) = 0_R$	by the distributive property.
$\therefore a = 0_R$ or $b - c = 0_R$	because integral domains do not have zero divisors.
$\therefore b - c = 0_R$	because $a \neq 0_R$
$\therefore b = c$	by addition

10) Let R be an integral domain with unity and let $a \in R$ be prime. Prove that a is also irreducible. (20 points)

Let R be an integral domain.

Assume $a \in R$ is prime.

Assume $a = xy$ for some $x, y \in R$.

$\therefore a \cdot 1_R = xy$	
$\therefore a \mid xy$	Definition of division
$\therefore a \mid x$ or $a \mid y$	Because a is prime.
wlog $a \mid x$	
$\therefore ak = x$ for some $k \in R$	Definition of division.
$\therefore xyk = x$	Plug in $a = xy$ into $ak = x$.
$\therefore yk = 1_R$	Cancellation from the fact that R is an integral domain.
$\therefore y \in R^*$	Definition of unit
$\therefore a$ is irreducible.	

11) Let R be a commutative ring with unity. Fix two elements $a, b \in R$. Prove that if $a = bt$ for some $t \in R$, then $\langle a \rangle \subseteq \langle b \rangle$. (20 points)

Assume $a = bt$ for some $t \in R$.
Suppose $x \in \langle a \rangle$
$\therefore x = ak$ for some $k \in R$.
$\therefore x = (bt)k = b(tk)$
$\therefore x \in \langle b \rangle$
$\langle a \rangle \subseteq \langle b \rangle$