Codename ____ Group Theory, Test 1

(Do not put your name on the test; write your name and codename on the code sheet)

1) Prove or disprove: $(\mathbb{Z},*)$ is a group where $a * b \coloneqq a + b - 1$

2) Find the order of $(20, 3)$ in $\mathbb{Z}_{99} \times \mathbb{Z}_{299}$ and justify your answer.

3) Prove that any infinite cyclic group has at most two generators.

4) Prove that $\mathbb{R} \times \mathbb{R} - \{(0,0)\}$ and $\mathbb{C} - \{0\}$ are not isomorphic. Here $\mathbb{R} \times \mathbb{R}$ has its operation defined by the direct product on the multiplicative group $\mathbb{R} - \{0\}$ while $\mathbb{C}$ has its operation defined by standard multiplication.

Codename ____ Group Theory, Sheet 2

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5) Let G and H be groups. Denote their operations as $\circ$ and $\star$ respectively. Denote their identities as e_G and e_H respectively.

Let $\varphi: G \to H$ be a homomorphism. Define the <u>kernel</u> of φ as $\ker(\varphi) \coloneqq \{g \in G | \varphi(g) = e_H\}$. Assume $\ker(\varphi) = \{e_G\}$. Show that φ is one-to-one.

6) Let $G = \{e, g, g^2, g^3, \dots, g^{n-1}\}$ be a finite cyclic group. Show that $|e| + |g| + |g^2| + \cdots + |g^n| > |G|$.

7) Let $G = \left\{e, g_1, g_2, g_3, \dots, g_{(n-1)}\right\}$ be a finite group. Show that $|e| + |g_1| + |g_2| + \cdots + |g_{n-1}| > |G|$.

Codename ____ Group Theory, Sheet 3

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8) Let G be a group. Assume there is a nonempty set $H \subseteq G$ such that H contains $a^{-1}b$ whenever $a, b \in H$. Show that H is a group.

9) Let G be a group and let H be a subgroup. For every $a \in G$, define $aHa^{-1} \coloneqq \{aha^{-1} | h \in H\}$. Show that aHa^{-1} is a subgroup of G.

10) Let G and H be multiplicative groups with an isomorphism φ from G to H. Show that $\psi: G \to H$ is a homomorphism where $\psi(x) \coloneqq \left(\varphi(x)\right)^2$

11) Consider the set of matrices $\mathbb{R}^{2\times 2}$ equipped with the standard matrix multiplication and $\mathbb{R}$ equipped with standard multiplication. We shall define the function $\varphi: \mathbb{R}^{2\times 2} \to \mathbb{R}$ via the equation below. Prove or disprove that this is a group homomorphism.

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab - cd$$