Codename ____ Group Theory, Test 1

(Do not put your name on the test; write your name and codename on the code sheet)

1) Prove or disprove: $(\mathbb{Z},*)$ is a group where $a * b := a + b - 1$

In order to show that $(\mathbb{Z},*)$ is a group, we must show that (0) the operation is defined on all of $\mathbb{Z}$ and is closed on $\mathbb{Z}$, (1) that $*$ is associative, (2) that there is an identity, and (3) that every element has an inverse.

First we note that $*$ is defined in terms of standard addition and subtraction of integers. Thus because addition and subtraction is defined on and closed on all of $\mathbb{Z}$, so $*$ is as well.

To show that $*$ is associative, we will show that $a * (b * c) = (a * b) * c$ for any arbitrary choice of integers a, b, c. To this end, we now fix these arbitrary integers $a, b, c \in \mathbb{Z}$ and do some algebra:

$$a * (b * c) = a * (b + c - 1) = a + (b + c - 1) - 1 = a + b + c - 2$$
$$= (a + b - 1) + c - 1 = (a * b) + c - 1 = (a * b) * c.$$

We will next show that $1 \in \mathbb{Z}$ is an identity. Fix an arbitrary integer $x \in \mathbb{Z}$. We now compute $1 * x$ and $x * 1$ and show that both of these come out to x:

$$1 * x = 1 + x - 1 = x$$
$$x * 1 = x + 1 - 1 = x.$$

We will next show $(\mathbb{Z},*)$ is closed under inverses. Fix an arbitrary $x \in \mathbb{Z}$. We will show that $x^{-1} = 2 - x$:

$$x * (2 - x) = x + (2 - x) - 1 = x + 2 - x - 1 = 2 - 1 = 1$$
$$(2 - x) * x = (2 - x) + x - 1 = 2 - x + x - 1 = 2 - 1 = 1.$$

Therefore, because we have proven each of the items $(0), (1), (2), (3)$ that define a group, indeed $(\mathbb{Z},*)$ is a group.

2) Find the order of $(20, 3)$ in $\mathbb{Z}_{99} \times \mathbb{Z}_{299}$ and justify your answer.

I intended $\mathbb{Z}_{100} \times \mathbb{Z}_{300}$...

The solution to this problem was presented by a student.

3) Prove that any infinite cyclic group has at most two generators.

The solution to this problem was presented by a student.

4) Prove that $\mathbb{R} \times \mathbb{R} - \{(0,0)\}$ and $\mathbb{C} - \{0\}$ are not isomorphic. Here $\mathbb{R} \times \mathbb{R}$ has its operation defined by the direct product on the multiplicative group $\mathbb{R} - \{0\}$ while $\mathbb{C}$ has its operation defined by standard multiplication.

The operation makes no sense… $\mathbb{R} \times \mathbb{R} - \{(0,0)\}$ is not a subset of $(\mathbb{R} - \{0\}) \times (\mathbb{R} - \{0\})$.

The solution to this problem was presented by a student.

5) Let G and H be groups. Denote their operations as $\circ$ and $\star$ respectively. Denote their identities as e_G and e_H respectively.
Let $\varphi: G \to H$ be a homomorphism. Define the kernel of φ as $\ker(\varphi) \coloneqq \{g \in G | \varphi(g) = e_H\}$. Assume $\ker(\varphi) = \{e_G\}$. Show that φ is one-to-one.

First note that to be one-to-one means that every output comes from at most one input. We express this precisely below:

"If $\varphi(x_1) = \varphi(x_2)$, then $x_1 = x_2$."

Assume that $\varphi(x_1) = \varphi(x_2)$. Then because H is a group, $\varphi(x_2)$ has an inverse, namely $\left(\varphi(x_2)\right)^{-1}$. Apply this to both sides of the equation from the right:

$$\varphi(x_1)\left(\varphi(x_2)\right)^{-1} = \varphi(x_2)\left(\varphi(x_2)\right)^{-1}.$$

Next we simplify this. By construction the right hand side is e_H. On the left hand side we use properties of homomorphisms to obtain

$$\varphi(x_1 x_2^{-1}) = e_H.$$

Now we note that the kernel of φ is $\{e_G\}$, and so in fact $x_1 x_2^{-1} = e_G$. Hence $x_1 = x_2$. Therefore φ is one-to-one.

6) Let $G = \{e, g, g^2, g^3, \dots, g^{n-1}\}$ be a finite cyclic group. Show that $|e| + |g| + |g^2| + \cdots + |g^n| > |G|$.

I intended $|e| + |g| + |g^2| + \cdots + |g^{n-1}| > |G|$.

The solution to this problem was presented by a student.

7) Let $G = \{e, g_1, g_2, g_3, \dots, g_{(n-1)}\}$ be a finite group. Show that $|e| + |g_1| + |g_2| + \cdots + |g_{n-1}| > |G|$.

I intended to assume $|e| + |g_1| + |g_2| + \cdots + |g_{n-1}| \geq |G|$.

The solution to this problem was presented by a student.

Codename ____ Group Theory, Sheet 3

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8) Let G be a group. Assume there is a nonempty set $H \subseteq G$ such that H contains $a^{-1}b$ whenever $a, b \in H$. Show that H is a group.

~~We will use the subgroup criterion and show that if $x, y \in H$, then also $xy^{-1} \in H$.~~

We will use the subgroup criterion and show that H contains the identity, that it is closed under inverses, and that it is closed under the operation.

First we show that H contains the identity. H is nonempty, so there is some $z \in H$. Hence by applying the given property, $e = z^{-1}z \in H$.

Next we show that H contains inverses. Suppose $z \in H$. Then choose $a = z$ and $b = e$ to get that $z^{-1} = z^{-1}e \in H$.

Lastly we show closure under the operation. Let $x, y \in H$. Choose $a = x^{-1}$ and $b = y$. Then we get

$$xy = (x^{-1})^{-1}y \in H.$$

Therefore H is a subgroup of G.

9) Let G be a group and let H be a subgroup. For every $a \in G$, define $aHa^{-1} := \{aha^{-1} | h \in H\}$. Show that aHa^{-1} is a subgroup of G.

We will use the subgroup criterion and show that aHa^{-1} is nonempty, closed under multiplication, and closed under inverses.

To see that aHa^{-1} is nonempty, note that $e \in H$. Hence $e = aea^{-1} \in H$.

To see that aHa^{-1} is closed under multiplication, take two arbitrary elements $x, y \in H$ and multiply them. Because $x, y \in H$ there are h_1, h_2 such that $x = ah_1a^{-1}, y = ah_2a^{-1}$ so that we get:

$$xy = ah_1a^{-1}ah_2a^{-1} = ah_1h_2a^{-1} \in H$$

To see that aHa^{-1} is closed under inverses, take an arbitrary element $x \in H$ and invert it. Again, we can write x in the proper form using some $h_3 \in H$:

$$x = ah_3a^{-1}.$$

Hence we see that:

$$x^{-1} = (ah_3a^{-1})^{-1} = (a^{-1})^{-1}h_3^{-1}a^{-1} = ah_3^{-1}a^{-1} \in H.$$

Therefore $aHa^{-1} \leq G$.

10) Let G and H be multiplicative groups with an isomorphism φ from G to H. Show that $\psi: G \to H$ is a homomorphism where $\psi(x) \coloneqq \left(\varphi(x)\right)^2$

The solution to this problem was presented by a student.

11) Consider the set of matrices $\mathbb{R}^{2\times 2}$ equipped with the standard matrix multiplication and $\mathbb{R}$ equipped with standard multiplication. We shall define the function $\varphi: \mathbb{R}^{2\times 2} \to \mathbb{R}$ via the equation below. Prove or disprove that this is a group homomorphism.

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab - cd$$

The solution to this problem was presented by a student.