Choose one of the problems below. Please provide scratch work/motivation/main ideas as well as a formal proof. (50+100 points)

1a) Let $G$ be a group. Let $I$ be some fixed index set, and $\{H_i | i \in I\}$ a collection of subgroups of $G$. That is, for each $i \in I$, $H_i \leq G$. Prove that

$$\bigcap_{i \in I} H_i \leq G.$$ 

1b) Let $G$ be a multiplicative group and fix two elements $a, b \in G$. Define the set below.

$$\langle a, b \rangle := \left\{ x_1^{y_1} x_2^{y_2} x_3^{y_3} \cdots x_n^{y_n} \middle| n \in \mathbb{Z}_{\geq 0}, x_i \in \{a, b\}, y_i \in \{-1, 1\} \right\}$$

Prove that

$$\langle a, b \rangle \leq G.$$ 

1c) Let $G$ be a group and $p$ be a prime. Assume that $|G| = p^2$. Show that $G$ is abelian.
Choose one of the problems below. Please provide scratch work/motivation/main idea as well as a formal proof. (50+100 points)

2a) Consider $\mathbb{Q} - \{0\}$ under multiplication and the normal subgroup $H = \left\{ \frac{a}{b} \mid a, b \text{ are both odd} \right\} \trianglelefteq \mathbb{Q}$. Now consider $\mathbb{Z}$ as a group under addition and show that:

$$\mathbb{Q}/H \cong \mathbb{Z}.$$

2b) Show that:

$$SL_2(\mathbb{R}) \leq GL_2(\mathbb{R}).$$

2c) Let $G$ and $H$ be groups with normal subgroups $A \trianglelefteq G$ and $B \trianglelefteq H$. Show that:

$$G \times H / A \times B \cong G/A \times H/B.$$
3) For each of the following, answer true or false. No justification required. (5 points each)

___ A group is a set.
___ A subgroup is a group.
___ Every subgroup of a cyclic group is normal.
___ If both of the left and right cosets of a subgroup partition the group, then the subgroup is a normal.
___ If a subgroup $H$ is normal in $G$, then the torsion subgroup of $G$ is finite.
___ Let $G$ be a fixed group, and denote the identity as $e$. Every group contains $e$.
___ “Assume $x$ is an arbitrary real number” fixes $x$ meaning that you no longer have control over what $x$ is in your proof.
___ Circular reasoning can be used to prove a theorem in group theory if and only if it is a result about nonassociative groups.
___ Up to isomorphism, there is a unique group of order 31.
___ Let $n$ be a number that is not prime. There is a group with order $n$ that is not cyclic.