Name ____ Test 2, Fall 2018

1) Prove that the sequence $\left\{\sqrt{\frac{1}{n}}\right\}$ converges as $n \to \infty$.

Let $\varepsilon > 0$.

Choose $N = \frac{1}{\varepsilon^2} + 1$

For all $n \geq N$ we have:

$$\sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{N}} = \sqrt{\frac{1}{\frac{1}{\varepsilon^2} + 1}} < \sqrt{\frac{1}{\frac{1}{\varepsilon^2}}} = \sqrt{\varepsilon^2} = \varepsilon$$

2) Assume that $\left\{\frac{2n+1}{n}\right\} \to 2$ as $n \to \infty$.

Prove that the sequence $\left\{\left(\frac{2n+1}{n}\right)^3 + 3\left(\frac{2n+1}{n}\right)^2 + 4\right\}$ converges as $n \to \infty$.

Let $a_n = \frac{2n+1}{n}$, then the assumption says that $\{a_n\} \to 2$.

From theorem T30, $\{f(a_n)\} \to f(2)$ for any polynomial $f(x)$. By taking $f(x) = x^3 + 3x^2 + 4$ this proves the result.

3) Assume that $\{a_n\}$ and $\{b_n\}$ are monotone increasing sequences. If $\forall_{n\in\mathbb{N}}(|a_n b_n| \leq 97)$.
Prove that the product sequence $\{a_n b_n\}$ also converges.

First note that $\{a_n b_n\}$ is also monotonic increasing:
$a_{n+1}b_{n+1} \geq a_n b_n$ because $a_{n+1} > a_n$ and $b_{n+1} > b_n$

The statement $\forall_{n\in\mathbb{N}}(|a_n b_n| \leq 97)$ says that $\{a_n b_n\}$ is bounded by 97. Therefore by the monotone convergence theorem, $\{a_n b_n\}$ converges.

NOTE: The above "proof" is incorrect because it has a mistake. In fact, the statement given in this problem in false. I realized this when going over the test solutions during class. A 14.2% extra credit bonus was given to anyone able to prove that the given was false, due the following class period.

4) Choose ONE of the problems below to complete.

(A) Prove that the interval $(2,5]$ is not compact.

(B) Prove that the interval $[1,7]$ is sequentially compact.

(A) $(2,5]$ is not closed because $\left\{2+\frac{1}{n}\right\} \to 2$, but $2 \notin (2,5]$. Therefore by theorem T37 $(2,5)$ is not compact.

(B) $[1,7]$ is a closed interval and is thus a closed set. It is bounded by 7. Therefore by theorem T37 it is sequentially compact.

5) Choose ONE of the problems below to complete.

(A) Prove that $f: (-\infty, -1) \cup (-1,1) \cup (1, \infty) \to \mathbb{R}$ given by $f(x) = \frac{2x+1}{x^2-1}$ is continuous.

(B) Prove that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is continuous.

(A) $f(x)$ is a quotient of two polynomials, and thus it is continuous by corollary C40.

(B) This problem takes a bit more work because we don't have a theorem for it. So we'll split it into three pieces, two of which are polynomials:

$$f(x) = \begin{cases} -x, x < 0 \\ 0, x = 0 \\ x, x > 0 \end{cases}$$

For $x < 0$ and $x > 0$ note that $f(x)$ is a polynomial and is thus continuous by C40. For $x = 0$ we take an arbitrary sequence $\{x_n\} \to 0$. In this case $\{|f(x_n)|\} = \{|x_n|\} \to 0$. Therefore $f(x_n) \to 0 = f(0)$ and so $f(x)$ is continuous.

6) Choose ONE of the problems below to complete.

(A) Assuming up to theorem T37 and that $f, g: D \to \mathbb{R}$ are continuous, prove that $f + g$ is continuous.

(B) Assuming up to theorem T40 prove theorem T41.

(A) Assume $x_0 \in D$ and that $\{x_n\} \to x_0$. Then by the assumption that f and g are continuous, we know that:

$$\{f(x_n)\} \to f(x_0)$$
$$\{g(x_n)\} \to g(x_0)$$

Therefore by theorem T29,

$$\{f(x_n) + g(x_n)\} \to f(x_0) + g(x_0) = (f + g)(x_0)$$

This shows that f and g are continuous.

(B) Assume $x_0 \in D$ and that $\{x_n\} \to x_0$. Then by the assumption that f is continuous at x_0, we know that:

$$\{f(x_n)\} \to f(x_0)$$

Now by the assumption that g is continuous at $f(x_0)$ we have that:

$$\{g(f(x_n))\} \to g(f(x_0)) = (g \circ f)(x_0)$$

This shows that $g \circ f$ is continuous at x_0.

True or False Problems

Each question is worth 0 points if left blank. If correct you gain points. If incorrect, you lose points. This means that if you do not know the answer, leaving it blank has the same expected point value as guessing.

T or F 7) A sequence can be convergent. – If for some $a \in \mathbb{R}$, $\forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n\geq N} (|a_n - a| < \varepsilon)$

T or F 8) A sequence can be bounded. – If $\exists_{M\in\mathbb{R}} \forall_{n\in\mathbb{N}} (|a_n| \leq M)$

T or F 9) A sequence can be continuous.

T or F 10) A sequence can be compact.

T or F 11) A sequence can be monotone. – If $\forall_{n\in\mathbb{N}} (a_{n+1} \geq a_n)$ or $\forall_{n\in\mathbb{N}} (a_{n+1} \leq a_n)$

T or F 12) A function can be convergent.

T or F 13) A function can be bounded. – If $\exists_{M\in\mathbb{R}} \forall_{x\in D} (|f(x)| \leq M)$

T or F 14) A function can be continuous. – If $\forall_{\{x_n\}\subseteq D, \{x_n\}\to x_0} \left(\{f(x_n)\} \to f(x_0)\right)$

T or F 15) A function can be compact.

T or F 16) A function can be monotone. – Technically true, but we did not define monotone functions.

T or F 17) A set of real numbers can be convergent.

T or F 18) A set of real numbers can be bounded. – If $\exists_{M\in\mathbb{R}} \forall_{x\in S} (|x| \leq M)$

T or F 19) A set of real numbers can be continuous.

T or F 20) A set of real numbers can be compact. – If Every open cover of S contains a finite subcover.

T or F 21) A set of real numbers can be monotone.

T or F 22) A real number can be convergent.

T or F 23) A real number can be bounded.

T or F 24) A real number can be continuous.

T or F 25) A real number can be compact.

T or F 26) A real number can be monotone.

27) Find the supremum of the set below.

$$\sup\left(\bigcup_{k=1}^{\infty}\left[-k, \frac{1}{k}\right)\right) = 1$$

$$\bigcup_{k=1}^{\infty}\left[-k, \frac{1}{k}\right) = [-1,1) \cup \left[-2, \frac{1}{2}\right) \cup \left[-3, \frac{1}{3}\right) \cup \cdots$$