Name ____ Test 3, Fall 2018

This test comes in four parts. Each part will be graded sequentially if the prerequisite conditions are met. You may answer as many or as few questions as you like. Take note of the following:

- In each section you cannot earn more points than the maximum score (No extra credit)
- Each section will not be counted unless the prerequisite is met (Start with Part 1)

Part	Number of questions	Points per question	Maximum Score	Prerequisite
1	10	15	59 (Cumulative 59)	None
2	5	10	20 (Cumulative 79)	59 points on part 1
3	4	5	10 (Cumulative 89)	20 points on part 2
4	6	2	11 (Cumulative 100)	10 points on part 3

Part 1

1) Let $f: \mathbb{R} \to \mathbb{R}$ be a function. What does it mean for f to be continuous? State the sequential definition.

2) Let $f: \mathbb{R} \to \mathbb{R}$ be a function. What does it mean for f to be continuous? State the $\varepsilon - \delta$ definition.

3) Let $f: [0,1] \to \mathbb{R}$ be a continuous function. Is it guaranteed that f has a maximum? Prove your answer.

4) Prove that there is a solution to the equation $x^9 + x^2 + 4 = 0$.

5) Let $f: \mathbb{R} \to \mathbb{R}$ be a function. What does the notation below mean? Formally state the definition.

$$\lim_{x \to a} f(x) = L$$

6) Formally state the definition of the derivative.

7) Use the definition of the derivative to compute the derivative of $f(x) = \sqrt{x+1}$ for all $x > 0$.

8) Explain the difference between a *maximum*, a *maximizer*, and a *supremum*.

9) Prove that $\{x^2 + 3x - 2 | 6 \leq x \leq 10\}$ is an interval.

10) Let $f: [0,1] \to \mathbb{R}$ be defined as below. Prove that $\underline{\int_a^b} f \geq 0$ and $\overline{\int_a^b} f \leq 1$

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Part 2

11) Prove that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous at $x = 5$ using the sequential definition.

12) Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = mx + b$ where $m, b \in \mathbb{R}$ are constants. Prove that f is uniformly continuous.

13) Prove that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not uniformly continuous.

14) What is integration? State the definitions of Lower Darboux sum, lower integral, and the integral itself.

15) Prove that the function $f: (0,1) \to \mathbb{R}$ given by $f(x) = 4x + 3$ has no maximum.

Part 3

16) Prove that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous at $x = 5$ using the $\varepsilon - \delta$ definition.

17) Let $f\colon [0,1] \to \mathbb{R}$ be defined as below. Prove that $\overline{\int_a^b f} \leq \frac{1}{2}$

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

18) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined below. Prove that f is differentiable at 0.

$$f(x) = \begin{cases} x^2, & x \leq 0 \\ x^3, & x > 0 \end{cases}$$

19) Below is a table of values. Run through 3 iterations of the bisection method and report your answer. Obviously, show your work.

x	$f(x)$
0	-432
1/32	-427
2/32	-412
3/32	-400
4/32	-382
5/32	-360
6/32	-333
7/32	-276
8/32	-200

x	$f(x)$
9/32	-196
10/32	-184
11/32	-178
12/32	-166
13/32	-164
14/32	-152
15/32	-109
16/32	-80

x	$f(x)$
17/32	-62
18/32	45
19/32	49
20/32	55
21/32	75
22/32	114
23/32	179
24/32	228

x	$f(x)$
25/32	316
26/32	330
27/32	332
28/32	336
29/32	430
30/32	469
31/32	486
1	495

Part 4

20) Let $a, b \in \mathbb{R}$ with $a < b$. Assume $f: (a, b) \to \mathbb{R}$ is monotonically increasing. Also assume f is bounded. Prove that the limit below exists.

$$\lim_{x \to a} f(x)$$

21) Let $f: \mathbb{R} \to \mathbb{R}$ be defined below. Prove that f is continuous at 0, using the $\varepsilon - \delta$ definition.

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ -x^2, & x \notin \mathbb{Q} \end{cases}$$

22) Let $f: \mathbb{R} \to \mathbb{R}$ be defined below. Prove that f is not differentiable anywhere.

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

23) Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ that not integrable anywhere. Then prove that it is not integrable on the domain $[a, b]$.

24) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. A *fixed point* is a point where the input and output of f are the same. Prove that f has a fixed point.

25) Let $f: \mathbb{Z} \to \mathbb{R}$ be a function. Show that f is continuous.