Name ____ Test 3, Fall 2018

This test comes in four parts. Each part will be graded sequentially if the prerequisite conditions are met. You may answer as many or as few questions as you like. Take note of the following:

- In each section you cannot earn more points than the maximum score (No extra credit)
- Each section will not be counted unless the prerequisite is met (Start with Part 1)

Part	Number of questions	Points per question	Maximum Score	Prerequisite
1	10	15	59 (Cumulative 59)	None
2	5	10	20 (Cumulative 79)	59 points on part 1
3	4	5	10 (Cumulative 89)	20 points on part 2
4	6	2	11 (Cumulative 100)	10 points on part 3

Part 1

1) Let $f: \mathbb{R} \to \mathbb{R}$ be a function. What does it mean for f to be continuous? State the sequential definition.

f is continuous at $x_0 \in \mathbb{R}$ If whenever $\{x_n\} \to x_0$, also $\{f(x_n)\} \to f(x_0)$. f is continuous if it is continuous for all $x_0 \in \mathbb{R}$.

2) Let $f: \mathbb{R} \to \mathbb{R}$ be a function. What does it mean for f to be continuous? State the $\varepsilon - \delta$ definition.

f is continuous at $x_0 \in \mathbb{R}$ if $\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x\in\mathbb{R}} (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)$. f is continuous if it is continuous for all $x_0 \in \mathbb{R}$.

3) Let $f: [0,1] \to \mathbb{R}$ be a continuous function. Is it guaranteed that f has a maximum? Prove your answer.

Yes! Note that $[0,1]$ is a closed interval, so by the extreme value theorem it has a maximum.

4) Prove that there is a solution to the equation $x^9 + x^2 + 4 = 0$.

Let $f(x) = x^9 + x^2 + 4$

$$f(-2) = -512 + 4 + 4 = -504 < 0$$
$$f(0) = 4 > 0$$

Therefore by the intermediate value theorem, f has a root.

5) Let $f: \mathbb{R} \to \mathbb{R}$ be a function. What does the notation below mean? Formally state the definition.

$$\lim_{x \to a} f(x) = L$$

This means that for all $\{x_n\}$ such that $\{x_n\} \to a$, it is true that $\{f(x_n)\} \to L$

6) Formally state the definition of the derivative.

The derivative of $f(x)$ at $x = a$ is:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

7) Use the definition of the derivative to compute the derivative of $f(x) = \sqrt{x+1}$ for all $x > 0$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$
$$= \lim_{h \to 0} \frac{(x+h+1) - (x+1)}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$
$$= \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h+1} + \sqrt{x+1}\right)} = \frac{1}{\left(\sqrt{x+1} + \sqrt{x+1}\right)} = \frac{1}{2\sqrt{x+1}}$$

8) Explain the difference between a *maximum*, a *maximizer*, and a *supremum*.

Let $f: D \to \mathbb{R}$ be some function. A maximum value of f is an element in the range that is larger than every other element in the range:

$$\forall_{x \in D}\left(f(x_0) \geq f(x)\right)$$

The maximizer is the input x_0 that gives the maximum above.

A supremum is the least upper bound of f, which is the maximum if such a thing exists. But even if the maximum does not exists, the supremum might.

9) Prove that $\{x^2 + 3x - 2 | 6 \leq x \leq 10\}$ is an interval.

Note that $[6,10]$ is an interval and $f(x) = x^2 + 3x - 2$ is a continuous function. Hence by theorem T44 its range, $\{x^2 + 3x - 2 | 6 \leq x \leq 10\}$, is an interval.

10) Let $f : [0,1] \to \mathbb{R}$ be defined as below. Prove that $\underline{\int_a^b} f \geq 0$ and $\overline{\int_a^b} f \leq 1$

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Choose the partition $P = \{[0,1]\}$. Then $L(f, P) = \inf(f(P)) = 0$ and $U(f, p) = \sup(f(P)) = 1$. Hence we have that $\underline{\int_a^b} f \geq 0$ and $\overline{\int_a^b} f \leq 1$.

Part 2

11) Prove that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous at $x = 5$ using the sequential definition.

Let $\{x_n\}$ be a sequence such that $\{x_n\} \to 5$.
Then by the product property of convergence: $\{x_n \cdot x_n\} \to 25$

$$\therefore \{f(x_n)\} = \{(x_n)^2\} \to 25 = f(5)$$

12) Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = mx + b$ where $m, b \in \mathbb{R}$ are constants. Prove that f is uniformly continuous.

Note that the definition of uniform continuity is:

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x,y\in\mathbb{R}} (|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$$

Or the sequential definition:
If $\{x_n\}$ and $\{y_n\}$ are sequences such that $\{x_n - y_n\} \to 0$, then $\{f(x_n) - f(y_n)\} \to 0$

Let $\varepsilon > 0$
Choose $\delta = \frac{\varepsilon}{m}$
Assume $|x - y| < \delta = \frac{\varepsilon}{m}$

$$\therefore |f(x) - f(y)| = |(mx + b) - (my + b)| = |m(x - y)| < m\delta = \varepsilon$$

13) Prove that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not uniformly continuous.

Let $\{x_n\} = \left\{n + \frac{1}{n}\right\}$ and $\{y_n\} = \left\{n - \frac{1}{n}\right\}$

$$\{x_n - y_n\} = \left\{n + \frac{1}{n} - n + \frac{1}{n}\right\} = \left\{\frac{2}{n}\right\} \to 0$$

However,

$$\{f(x_n) - f(y_n)\} = \left\{\left(n + \frac{1}{n}\right)^2 - \left(n - \frac{1}{n}\right)^2\right\} = \left\{n^2 + 2 + \frac{2}{n^2} - n^2 + 2 - \frac{2}{n^2}\right\} \to 4 \neq 0$$

14) What is integration? State the definitions of Lower Darboux sum, lower integral, and the integral itself.

The lower Darboux sum, written $L(f, P)$ is the summation of the infimum of $f(x)$ on each partition interval: If $P = \{[x_0, x_1], \ldots . [x_{n-1}, x_n]\}$, then:

$$L(f, P) = \sum_{k=1}^{n} \inf_{x \in [x_{k-1}, x_k]} \big(f(x)\big) \cdot (x_k - x_{k-1})$$

The lower integral, written $\underline{\int_a^b f}$, is the supremum of all the lower Darboux sums, considered over all partitions:

$$\underline{\int_a^b} f = \sup_P \big(L(f, P)\big)$$

If the lower and upper integrals are equal, we define the integral as their value:

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$$

Otherwise the integral does not exist.

15) Prove that the function $f: (0,1) \to \mathbb{R}$ given by $f(x) = 4x + 3$ has no maximum.

Suppose that y_0 is the maximum of f. Then there is some $x_0 \in (0,1)$ such that $f(x_0) = y_0$. However, because f is monotonically increasing, write $x_2 = \frac{x_0+1}{2} < 2$. Thus

$$f(x_2) > f(x_0) \text{ because } x_2 > x_0.$$

Part 3

16) Prove that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous at $x = 5$ using the $\varepsilon - \delta$ definition.

Let $\varepsilon > 0$

Choose $\delta = \min(7, \frac{\varepsilon}{17})$

Assume $|x - 5| < \delta$. Then:

$$|f(x) - f(5)| = |x^2 - 25| = |(x-5)(x+5)| < \delta|x+5| \leq 17\delta \leq \varepsilon$$

17) Let $f: [0,1] \to \mathbb{R}$ be defined as below. Prove that $\overline{\int_a^b f} \leq \frac{1}{2}$

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Let's take a sequence of partitions with:

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots \frac{n-1}{n}, 1\right\}$$

$$U(f, P_n) = \sum_{k=1}^{n} \frac{k}{n} \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{n(n+1)}{2n^2} \to \frac{1}{2}$$

Therefore $\overline{\int_a^b f} \leq \frac{1}{2}$

18) Let $f: \mathbb{R} \to \mathbb{R}$ be defined below. Prove that f is differentiable at 0.

$$f(x) = \begin{cases} x^2, & x \leq 0 \\ x^3, & x > 0 \end{cases}$$

We must show that the limit below exists.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

That means that

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

However, that is easy:

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(0+h)^2 - 0^2}{h} = \frac{d}{dx} x^2 \bigg|_{x=0} = 0$$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(0+h)^3 - 0^3}{h} = \frac{d}{dx} x^3 \bigg|_{x=0} = 0$$

19) Below is a table of values. Run through 3 iterations of the bisection method and report your answer. Obviously, show your work.

x	$f(x)$
0	-432
1/32	-427
2/32	-412
3/32	-400
4/32	-382
5/32	-360
6/32	-333
7/32	-276
8/32	-200

x	$f(x)$
9/32	-196
10/32	-184
11/32	-178
12/32	-166
13/32	-164
14/32	-152
15/32	-109
16/32	-80

x	$f(x)$
17/32	-62
18/32	45
19/32	49
20/32	55
21/32	75
22/32	114
23/32	179
24/32	228

x	$f(x)$
25/32	316
26/32	330
27/32	332
28/32	336
29/32	430
30/32	469
31/32	486
1	495

$a_0 = 0; f(a_0) = -432$ and $b_0 = 1; f(b_0) = 495. \frac{a_0+b_0}{2} = \frac{16}{32}$ and $f\left(\frac{16}{32}\right) = -80$

$a_1 = \frac{16}{32}; f(a_1) = -80$ and $b_1 = 1; f(b_1) = 495. \frac{a_1+b_1}{2} = \frac{24}{32}$ and $f\left(\frac{24}{32}\right) = 228$

$a_2 = \frac{16}{32}; f(a_2) = -80$ and $b_2 = \frac{24}{32}; f(b_2) = 228.$

Therefore there is a root somewhere between $\frac{16}{32}$ and $\frac{24}{32}$.

Part 4

20) Let $a, b \in \mathbb{R}$ with $a < b$. Assume $f: (a, b) \to \mathbb{R}$ is monotonically increasing. Also assume f is bounded. Prove that the limit below exists.

$$\lim_{x \to a} f(x)$$

21) Let $f: \mathbb{R} \to \mathbb{R}$ be defined below. Prove that f is continuous at 0, using the $\varepsilon - \delta$ definition.

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ -x^2, & x \notin \mathbb{Q} \end{cases}$$

22) Let $f: \mathbb{R} \to \mathbb{R}$ be defined below. Prove that f is not differentiable anywhere.

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

23) Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ that not integrable anywhere. Then prove that it is not integrable on the domain $[a, b]$.

24) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. A *fixed point* is a point where the input and output of f are the same. Prove that f has a fixed point.

25) Let $f: \mathbb{Z} \to \mathbb{R}$ be a function. Show that f is continuous.