Name ____ Test 1, Fall 2020

This test comes in four parts. You may answer as many or as few questions as you like. Take note of the following:

- There is no partial credit, you earn points only for what you have mastered.
- Credit is given for correct answers, or *nearly* correct answers.
  (I won't split hairs on minor mistakes)
- There are more problems per part than is required for the maximum score
  (Balances out no partial credit)
- In each section you cannot earn more points than the maximum score
  (No extra credit)
- Please write on the blank paper provided. You may use multiple sheets if necessary. Please start each part on a new sheet (as I will be separating them into parts to grade in batches)

Part	Number of questions	Points per question	Maximum Score
1	10	15	59 (Cumulative 59)
2	6	10	20 (Cumulative 79)
3	5	5	10 (Cumulative 89)
4	4	3	11 (Cumulative 100)

Part 1

1) Give an example of a monotone sequence that does not converge.

2) Give an example of a set that has a supremum, but not a maximum.

3) Give an example of a set that does not have a supremum.

4) True or false and why? Every closed and bounded set is compact.

5) What is $|x|$? State the definition.

6) What does it mean for a set S to be sequentially compact? State the definition.

7) Let $\{a_n\}$ be a real sequence. What does it mean for $\{a_n\}$ to converge to a? State the sequential definition.

8) What does it mean for a set S to be compact? State the definition.

9) What is the definition of the universal quantifier, ∀? State the definition.

10) What is the infimum? State the definition of $\inf(S)$.

Part 2

11) Assume that $\{x_n\} \to 5$ and $\{y_n\} \to 2$. Prove that $\{3x_n + 2y_n\} \to 19$.

12) Give an example of an open cover of $\mathbb{R}$ that does not have a finite subcover.

13) Let $b > 0$ and assume $|x - b| < \frac{4}{5}|b|$. Prove that $x > \frac{b}{5}$

14) Prove that Prove that $\left\{\frac{1}{(n+2)^3} + 1\right\} \to 1$

15) Prove that the interval $(2,5]$ is not compact.

16) Prove that the interval $[1,7]$ is sequentially compact.

Part 3

17) Prove that Prove that $\left\{\frac{1}{(n+2)^3} + 1\right\} \to 1$ using the ε definition of convergence.

18) Prove that $\sqrt{2}$ is irrational

19) Given a real number a, define $S := \{x \in \mathbb{Q} : x < a\}$. Prove that $a = \sup(S)$

20) Let $\{a_n\}$ be a sequence that converges to a and $\{b_n\}$ a sequence. Assume that there is an index N such that $a_n = b_n$ for all $n \geq N$. Prove that $\{b_n\} \to a$.

21) Prove that the set $[5, \infty)$ is closed.

Part 4

22) Let $\{a_n\}$ be a sequence that converges to a and $\{b_n\}$ a sequence. Assume that there is an index N such that $a_n = b_n$ for all $n \geq N$. Use the definition of convergence to prove that $\{b_n\}$ converges.

23) Assume $\{a_n\}$ is monotone. Prove that $\{a_n\}$ converges if and only if $\{a_n^2\}$ converges.

24) Assume that $|a| < 1$ and $\{a_n\} \to a$. Prove that $\{a_n^n\} \to 0$

25) Let A and B be compact sets. Prove that $A \cup B$ is compact.