Name ____ Test 1, Fall 2020

This test comes in four parts. You may answer as many or as few questions as you like. Take note of the following:

- There is no partial credit, you earn points only for what you have mastered.
- Credit is given for correct answers, or *nearly* correct answers.
  (I won't split hairs on minor mistakes)
- There are more problems per part than is required for the maximum score
  (Balances out no partial credit)
- In each section you cannot earn more points than the maximum score
  (No extra credit)
- Please write on the blank paper provided. You may use multiple sheets if necessary. Please start each part on a new sheet (as I will be separating them into parts to grade in batches)

Part	Number of questions	Points per question	Maximum Score
1	10	15	59 (Cumulative 59)
2	6	10	20 (Cumulative 79)
3	5	5	10 (Cumulative 89)
4	4	3	11 (Cumulative 100)

Part 1

1) Give an example of a monotone sequence that does not converge.

We know that any monotone and bounded sequence converges (in fact to its sup). So you'll need an unbounded sequence, such as $\{n\}_{n=1}^{\infty}$ or $\{3n^2 + 2\}_{n=1}^{\infty}$.

2) Give an example of a set that has a supremum, but not a maximum.

If the set is unbounded above, it would not have a sup (well, ∞). So choose a set that is bounded, but yet still has no maximum. Perhaps, say, $(0,1)$.

3) Give an example of a set that does not have a supremum.

We know any bounded set has a supremum, so you'll need to choose something unbounded above. Say, $[0, \infty)$ or just all of $\mathbb{R}$.

4) True or false and why? Every closed and bounded set is compact.

True, look at theorem T41.

5) What is $|x|$? State the definition.

$$|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

6) What does it mean for a set S to be sequentially compact? State the definition.

S is sequentially compact if every sequence in S has a convergent subsequence. (it must converge in S).

7) Let $\{a_n\}$ be a real sequence. What does it mean for $\{a_n\}$ to converge to a? State the sequential definition.

For each tolerance ε, beyond some point N, a_n is within ε of a. That is:

$$\forall_{\varepsilon>0}\exists_{N\in\mathbb{N}}\forall_{n\geq N}(|a_n - a| < \varepsilon)$$

8) What does it mean for a set S to be compact? State the definition.

A set S is compact if every open cover of S has a finite subcover.

9) What is the definition of the universal quantifier, ∀? State the definition.

Let $S(x)$ be a statement, given any value of x. The universal quantifier ∀ quantifies the statement by making a statement is that true if and only if $S(x)$ is true for every x:

$$\forall_x\big(S(x)\big) \text{ is true if and only if } S(x) \text{ is true for each } x$$

10) What is the infimum? State the definition of $\inf(S)$.

$\inf(S)$ is the greatest lower bound of S.

Part 2

11) Assume that $\{x_n\} \to 5$ and $\{y_n\} \to 2$. Prove that $\{3x_n + 2y_n\} \to 19$.

$\{3x_n\} \to 3 \cdot 5 = 15$ by lemma L24.
$\{2y_n\} \to 2 \cdot 2 = 4$ by lemma L24
$\{3x_n + 2y_n\} \to 15 + 4 = 19$ by theorem T23

12) Give an example of an open cover of $\mathbb{R}$ that does not have a finite subcover.

$\{\dots, (-1,1), (0,2), (1,3), (2,4), \dots\}$

13) Let $b > 0$ and assume $|x - b| < \frac{4}{5}|b|$. Prove that $x > \frac{b}{5}$

Choosing $d = \frac{4}{5}|b|$ in theorem T14 we get:

$$-\frac{4}{5}|b| \leq x - b \leq \frac{4}{5}|b|$$

Just consider the left half, $-\frac{4}{5}|b| \leq x - b$, and add b to both sides to get:

$$\frac{|b|}{5} \leq x$$

$b > 0$, so this can be written as:

$$\frac{b}{5} \leq x$$

Okay technically we wanted strict inequality. Because the original inequality was strict, it turns out it will follow all the way through to get $\frac{b}{5} < x$.

14) Prove that Prove that $\left\{\frac{1}{(n+2)^3} + 1\right\} \to 1$

$\left\{\frac{1}{n}\right\} \to 0$ by T9
$\left\{\frac{1}{n+2}\right\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=3}^{\infty}$, so $\left\{\frac{1}{n+2}\right\} \to 0$ as well.
$\left\{\frac{1}{(n+2)^3}\right\} = \left\{\left(\frac{1}{n+2}\right)^3\right\} \to 0^3 = 0$ by theorem T30.
$\{1\} \to 1$, obviously.
$\left\{\frac{1}{(n+2)^3} + 1\right\} \to 0 + 1 = 1$ by theorem T23.

15) Prove that the interval $(2,5]$ is not compact.

The following open interval cover does not have a finite subcover:

$$\{(3,8),(2.1,4),(2.01,4),(2.001,4),\ldots\}$$

16) Prove that the interval $[1,7]$ is sequentially compact.

It is closed and bounded, so by T41 it is sequentially compact.

Part 3

17) Prove that Prove that $\left\{\frac{1}{(n+2)^3}+1\right\} \to 1$ using the ε definition of convergence.

Let $\varepsilon > 0$ and choose $N = \left\lceil \sqrt[3]{\frac{1}{\varepsilon}} \right\rceil$. Then we obtain for all $n \geq N$:

$$\left|\frac{1}{(n+2)^3}+1-1\right| = \left|\frac{1}{(n+2)^3}\right| = \frac{1}{(n+2)^3} < \frac{1}{n^3} \leq \frac{1}{N^3} = \frac{1}{\left(\left\lceil \sqrt[3]{\frac{1}{\varepsilon}} \right\rceil\right)^3} \leq \frac{1}{\left(\sqrt[3]{\frac{1}{\varepsilon}}\right)^3} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

Thus $\left\{\frac{1}{(n+2)^3}+1\right\} \to 1$

18) Prove that $\sqrt{2}$ is irrational

Assume $\sqrt{2} \in \mathbb{Q}$. Then we can write $\sqrt{2} = \frac{p}{q}$ and wlog assume $\gcd(p,q) = 1$.

$\therefore q\sqrt{2} = p$
$\therefore 2q^2 = p^2$
$\therefore p^2$ is even
$\therefore p$ is even
Write $p = 2k$ for some $k \in \mathbb{Z}$
$\therefore 2q^2 = (2k)^2 = 4k^2$
$\therefore q^2 = 2k^2$
$\therefore q^2$ is even
$\therefore q$ is even
This is a contradiction with the fact that $\gcd(p,q) = 1$, so $\sqrt{2} \notin \mathbb{Q}$.

19) Given a real number a, define $S := \{x \in \mathbb{Q} : x < a\}$. Prove that $a = \sup(S)$

By the of definition S, a an upper bound. Suppose that b is a smaller upper bound. That is, $b < a$ and $x < b$ for all $x \in S$. However, (b,a) contains a rational number, say c, by theorem T13. This is a contradiction because $c \in \mathbb{Q}$ and $c < a$. Hence b was not an upper bound for S. Therefore $a = \sup(S)$.

20) Let $\{a_n\}$ be a sequence that converges to a and $\{b_n\}$ a sequence. Assume that there is an index N such that $a_n = b_n$ for all $n \geq N$. Prove that $\{b_n\} \to a$.

Consider the sequence $\{b_n - a_n\}$. Choosing $C = 0$ and the sequence $\{0\}$, lemma L21 tells us that $\{b_n - a_n\} \to 0$ because $|b_n - a_n| = |a_n - a_n| = 0 \leq 0$ for all $n \geq N$. Then apply T23 to $\{b_n - a_n\}$ and $\{a_n\}$ to obtain:

$$\{b_n\} = \{b_n - a_n + a_n\} \to 0 + a = a$$

21) Prove that the set $[5, \infty)$ is closed.

Let $\{x_n\}$ be a sequence in $[5, \infty)$ and assume that $\{x_n\} \to x \in \mathbb{R}$. Assume for the purpose of later contradiction that $x < 5$. Choose $\varepsilon = \frac{5-x}{2}$. Then by convergence there is some $N \in \mathbb{N}$ such that

$$x_n \in \left(x - \frac{5-x}{2}, x + \frac{5-x}{2}\right)$$

for all $n \geq N$. Note that $x + \frac{5-x}{2} < 5$ (why?), so $x_n \notin [5, \infty)$ which is a contradiction. Hence $x \geq 5$, so $[5, \infty)$ is closed.

Part 4

22) Let $\{a_n\}$ be a sequence that converges to a and $\{b_n\}$ a sequence. Assume that there is an index N such that $a_n = b_n$ for all $n \geq N$. Use the definition of convergence to prove that $\{b_n\}$ converges.

Let $\varepsilon > 0$. We know that because $\{a_n\} \to a$, there is some $N_2 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N_2$. If we consider $N_3 = \max(N, N_2)$, then we see that $|b_n - a| = |a_n - a| < \varepsilon$ for all $n \geq N_3$. Thus we have $\{b_n\} \to a$.

23) Assume $\{a_n\}$ is monotone. Prove that $\{a_n\}$ converges if and only if $\{a_n^2\}$ converges.

The forward direction is a trivial consequence of T30 taking $f(x) = x^2$. The backward direction takes more work.

Assume $\{a_n^2\}$ converges. Because $\{a_n\}$ is monotone, $\{a_n^2\}$ is also monotone. Thus $\{a_n^2\}$ is bounded by theorem T35. Hence $\{a_n\}$ is also bounded. Then again by T35, $\{a_n\}$ is converges.

24) Assume that $|a| < 1$ and $\{a_n\} \to a$. Prove that $\{a_n^n\} \to 0$

Let us create a sequence of sequences. $\{\{a_n^m\}_{n=1}^{\infty}\}_{m=1}^{\infty}$. For each fixed m, T30 tells us that $\{a_n^m\}_{n=1}^{\infty} \to a^m$. However, note that $\{a^m\} \to 0$. Hence $\{a_n^n\}_{n=0}^{\infty} \to 0$.

25) Let A and B be compact sets. Prove that $A \cup B$ is compact.

By T41 both A and B are closed and bounded. Because they are both bounded, $A \cup B$ is obviously also bounded (By the larger of the two bounds). For closedness, let $\{a_n\}$ be a sequence in $A \cup B$ and assume $\{a_n\} \to a \in \mathbb{R}$. Either $\{a_n\}$ has infinitely many terms in A, or it has infinitely many terms in B. Assume wlog that it has infinitely many terms in A, and consider the subsequence $\{a_{n_k}\}$ of those terms just in A. Because it is a subsequence of $\{a_n\}$, it converges to the same thing: $\{a_{n_k}\} \to a$. However, because A is closed and $\{a_{n_k}\}$ is in A, the limit, a, is in A. That is, $a \in A$. Therefore $A \cup B$ is closed, and together with boundedness we see that $A \cup B$ is compact.

Or a direct proof:
Let $\{I_n\}$ be an open interval cover of $A \cup B$. Then it is simultaneously an open interval covers for A and for B. Hence there are finite subcovers $\{I_n\}_{n=1}^{m_1}$ and $\{I_n\}_{n=1}^{m_2}$ that cover A and B respectively. Hence if we take the union of these two finite sets, we get a finite open subcover of $A \cup B$:

$$\{I_n\}_{n=1}^{\max(m_1, m_2)}$$