Name ____ Test 2, Fall 2020

This test comes in four parts. You may answer as many or as few questions as you like. Take note of the following:

- There is no partial credit, you earn points only for what you have mastered.
- Credit is given for correct answers, or *nearly* correct answers.
  (I won't split hairs on minor mistakes)
- There are more problems per part than is required for the maximum score
  (Balances out no partial credit)
- In each section you cannot earn more points than the maximum score
  (No extra credit)
- Please write on the blank paper provided. You may use multiple sheets if necessary. Please start each part on a new sheet (as I will be separating them into parts to grade in batches)

Part	Number of questions	Points per question	Maximum Score
1	10	15	59 (Cumulative 59)
2	6	10	20 (Cumulative 79)
3	5	5	10 (Cumulative 89)
4	5	3	11 (Cumulative 100)

Part 1

1) What does it mean for a function $f: D \to \mathbb{R}$ to have a maximum? State the definition.

2) What does it mean for a function $f: D \to \mathbb{R}$ to be uniformly continuous? State the definition.

3) State the $\varepsilon - \delta$ criterion of uniform continuity.

4) True or false and why: A one-to-one function is necessarily strictly monotonic.

5) True or false and why: If $f: [0,1) \to \mathbb{R}$ and $g: [1,2] \to \mathbb{R}$ are continuous, then the function below is also continuous.

$$h(x) \coloneqq \begin{cases} f(x), \text{ if } x < 1 \\ g(x), \text{ if } x \geq 1 \end{cases}$$

6) True or false and why: $f: [5,12] \to \mathbb{R}$ given by $f(x) = 3x^2$ is monotonic.

7) True or false and why: If $f: [0,1] \to \mathbb{R}$ is continuous at both $x = 0$ and $x = 1$, then it is continuous.

8) True or false and why: If I is an interval, and $f: I \to \mathbb{R}$ is a function, then f^{-1} exists and is continuous.

9) What does it mean for a function $f: D \to \mathbb{R}$ to be monotonic? State the definition.

10) True or false and why: Every continuous function $f: [0,1] \to \mathbb{R}$ has a maximum.

Part 2

11) Show that the function $f(x) = x^2 + 3$ is continuous at $x = 4$ using the sequential definition.

12) True or false and why: Let $f: [0,1] \to \mathbb{R}$ be a continuous function such that $f(x) > 2$ and $0 \leq x < 1$. It is necessarily the case that $f(1) > 2$.

13) Prove that the function $f: [0,5] \to \mathbb{R}$ given by $f(x) = 2x^2 + 3$ is uniformly continuous.

14) True or false and why: $f: [0,2] \to \mathbb{R}$ given by $f(x) = 2x^2 + 3$ has a maximum.

15) True or false and why: A strictly increasing function is necessarily one-to-one.

16) Let $f: (0,1) \to \mathbb{R}$ be a decreasing function such that $f\big((0,1)\big)$ is an interval. Prove that f is continuous.

Part 3

This new definition is needed for problems 17 and 18.
A function $f: D \to \mathbb{R}$ is called Lipschitz if there is a $C > 0$ such that for all $u, v \in D$:

$$|f(u) - f(v)| \leq C|u - v|$$

17) Prove that a Lipschitz function is also continuous.

18) Prove that the function $f: [0,1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is *not* a Lipschitz function.

19) Show that the function $f(x) = \begin{cases} x^2, \text{if } x < 1 \\ 2x - 1, \text{if } x \geq 1 \end{cases}$ is continuous at $x_0 = 1$.

20) Let S be a set that is not sequentially compact that contains no unbounded sequences. Prove that there is a sequence in S that converges to a point not in S.

21) Prove that $f(x) = x^2$ is continuous at $x = 5$ using the $\varepsilon - \delta$ criterion.

Part 4

22) Let $f: D \to \mathbb{R}$ be a function. A point such that $f(x) = x$ is called a fixed point. If the function $g: [-1,1] \to \mathbb{R}$ is continuous and $g(-1) > -1$ and $f(1) < 1$, prove that g has a fixed point.

23) Using the $\varepsilon - \delta$ criterion, prove that $f(x) = x^3$ is continuous at an arbitrary point x_0.

24) State what it means to be periodic, then prove that a periodic function $f: \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

25) Let $f: [a, b] \to \mathbb{R}$ be continuous and one-to-one. Assume $f(a) < f(b)$ and $c \in [a, b]$.
Prove that $c < f(b)$.

26) Prove that $f: \{0\} \to \mathbb{R}$ is continuous at 0, regardless of how f is defined.