Name ____ Test 2, Fall 2020

This test comes in four parts. You may answer as many or as few questions as you like. Take note of the following:

- There is no partial credit, you earn points only for what you have mastered.
- Credit is given for correct answers, or *nearly* correct answers.
  (I won't split hairs on minor mistakes)
- There are more problems per part than is required for the maximum score
  (Balances out no partial credit)
- In each section you cannot earn more points than the maximum score
  (No extra credit)
- Please write on the blank paper provided. You may use multiple sheets if necessary. Please start each part on a new sheet (as I will be separating them into parts to grade in batches)

Part	Number of questions	Points per question	Maximum Score
1	10	15	59 (Cumulative 59)
2	6	10	20 (Cumulative 79)
3	5	5	10 (Cumulative 89)
4	5	3	11 (Cumulative 100)

Part 1

1) What does it mean for a function $f: D \to \mathbb{R}$ to have a maximum? State the definition.

$f(x_0)$ is a maximum if $f(x_0) \geq f(x)$ for all $x \in D$.

2) What does it mean for a function $f: D \to \mathbb{R}$ to be uniformly continuous? State the definition.

If $\{u_n\}$ and $\{v_n\}$ are sequences such that $\{u_n - v_n\} \to 0$, then $\{f(u_n) - f(v_n)\} \to 0$

3) State the $\varepsilon - \delta$ criterion of uniform continuity.

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{u,v\in D} (|u - v| < 0 \Rightarrow |f(u) - f(v)| < 0)$$

4) True or false and why: A one-to-one function is necessarily strictly monotonic.

False. Consider a discontinuous function which jumps around a lot. For instance, $y = x$ when $x \in \mathbb{Q}$ and $y = -x$ when $x \notin \mathbb{Q}$.

5) True or false and why: If $f: [0,1) \to \mathbb{R}$ and $g: [1,2] \to \mathbb{R}$ are continuous, then the function below is also continuous.

$$h(x) \coloneqq \begin{cases} f(x), \text{ if } x < 1 \\ g(x), \text{ if } x \geq 1 \end{cases}$$

False. There's no reason they have to meet up. For example if $f(x) = 2x$ and $g(x) = 3x$.

6) True or false and why: $f: [5,12] \to \mathbb{R}$ given by $f(x) = 3x^2$ is monotonic.

True. Note that $x > 0$ in the domain, so if $x_1 > x_2$, then $x_1^2 > x_2^2$.

7) True or false and why: If $f: [0,1] \to \mathbb{R}$ is continuous at both $x = 0$ and $x = 1$, then it is continuous.

No… lol…. It could have a break somewhere in between, like at $x = 0.5$. Such as:

$$f(x) = \begin{cases} 2, \text{ if } x < 0.5 \\ 3, \text{ if } x \geq 0.5 \end{cases}$$

8) True or false and why: If I is an interval, and $f: I \to \mathbb{R}$ is a function, then f^{-1} exists and is continuous.

False. Consider for instance $f(x) = 5$ on $I = [0,1]$. It doesn't even have an inverse.

9) What does it mean for a function $f: D \to \mathbb{R}$ to be monotonic? State the definition.

It means that f is either increasing or decreasing.
(Technically also either strictly increasing or strictly increasing, but those are included in increasing or decreasing above.)

10) True or false and why: Every continuous function $f: [0,1] \to \mathbb{R}$ has a maximum.

True. This is an application of the extreme value theorem.

Part 2

11) Show that the function $f(x) = x^2 + 3$ is continuous at $x = 4$ using the sequential definition.

Let $\{u_n\}$ be a sequence such that $\{u_n\} \to 4$. Then by the polynomial property of convergence (T30), $\{u_n^2 + 3\} \to 4^2 + 3 = 19$. Because $f(4) = 19$, f is continuous at $x = 4$.

12) True or false and why: Let $f: [0,1] \to \mathbb{R}$ be a continuous function such that $f(x) > 2$ and $0 \leq x < 1$. It is necessarily the case that $f(1) > 2$.

This is false, it could be way up and away from 2, such as $f(x) = x + 3$.

13) Prove that the function $f: [0,5] \to \mathbb{R}$ given by $f(x) = 2x^2 + 3$ is uniformly continuous.

We know that f is continuous because it is a polynomial. By T51, f is uniformly continuous on $[0,5]$.

14) True or false and why: $f: [0,2] \to \mathbb{R}$ given by $f(x) = 2x^2 + 3$ has a maximum.

We know that f is continuous because it is a polynomial. By T47, f attains a maximum.

15) True or false and why: A strictly increasing function is necessarily one-to-one.

True. $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ because it is strictly increasing. In particular, if $u \neq v$, assume wlog $u > v$. Hence $f(u) > f(v)$.

16) Let $f: (0,1) \to \mathbb{R}$ be a decreasing function such that $f\big((0,1)\big)$ is an interval. Prove that f is continuous.

This is a direct result of T53 because f is monotone.

Part 3

This new definition is needed for problems 17 and 18.

A function $f: D \to \mathbb{R}$ is called <u>Lipschitz</u> if there is a $C > 0$ such that for all $u, v \in D$:

$$|f(u) - f(v)| \leq C|u - v|$$

17) Prove that a Lipschitz function is also continuous.

Let $\varepsilon > 0$

Choose $\delta = \frac{\varepsilon}{C}$

Let $x_1, x_2 \in D$ and assume $|x_1 - x_2| < \delta$.

$$|f(x_1) - f(x_2)| \leq C|x_1 - x_2| \leq C\delta \leq C\frac{\varepsilon}{C} = \varepsilon$$

Thus f is uniformly continuous and therefore continuous.

18) Prove that the function $f: [0,1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is *not* a Lipschitz function.

Let $C > 0$. We need to find points u, v such that $|f(u) - f(v)| > C|u - v|$. Take $u = 0$ and $v = \frac{1}{2C^2}$.

$$|f(u) - f(v)| = \left|0 - \sqrt{\frac{1}{2C^2}}\right| = \sqrt{\frac{1}{2C^2}} = \frac{1}{\sqrt{2}C} > \frac{1}{2C} = C \cdot \frac{1}{2C^2} = C|v|$$

19) Show that the function $f(x) = \begin{cases} x^2, \text{if } x < 1 \\ 2x - 1, \text{if } x \geq 1 \end{cases}$ is continuous at $x_0 = 1$.

Let $\{u_n\} \to 1$. If beyond some point $u_n < 1$, then $\{f(u_n)\} = \{u_n^2\} \to 1$. On the other hand if beyond some point $u_n > 1$, then $\{f(u_n)\} = \{2u_n - 1\} \to 1$. Hence $\{f(u_n)\} \to 1$. Lastly because $f(1) = 1$, it is thus continuous.

20) Let S be a set that is not sequentially compact that contains no unbounded sequences. Prove that there is a sequence in S that converges to a point not in S.

21) Prove that $f(x) = x^2$ is continuous at $x = 5$ using the $\varepsilon - \delta$ criterion.

Let $\varepsilon > 0$.

Choose $\delta = \min\left(\frac{\varepsilon}{11}, 1\right)$.

Let $x \in \mathbb{R}$ and assume $|x - 5| < \delta$.

$$|x^2 - 5^2| = |x - 5| \cdot |x + 5| \leq \delta|x + 5| \leq \delta(|x| + |5|) \leq \delta(6 + 5) = 11\delta < \frac{11\varepsilon}{11} = \varepsilon$$

Part 4

22) Let $f: D \to \mathbb{R}$ be a function. A point such that $f(x) = x$ is called a fixed point. If the function $g: [-1,1] \to \mathbb{R}$ is continuous and $g(-1) > -1$ and $f(1) < 1$, prove that g has a fixed point.

Consider the function $h(x) = x - g(x)$. We then have:

$$h(1) = 1 - g(1) < 1 - 1 = 0$$
$$h(-1) = -1 - g(-1) > -1 - (-1) = 0$$

Hence by the intermediate value theorem there is a point $c \in [-1,1]$ such that $h(c) = 0$. This c is a fixed point:

$$0 = h(c) = x - g(x)$$
$$g(x) = x$$

23) Using the $\varepsilon - \delta$ criterion, prove that $f(x) = x^3$ is continuous at an arbitrary point x_0.

Let $x_0 \in \mathbb{R}$
Let $\varepsilon > 0$
Choose $\delta = \min\left(1, \frac{\varepsilon}{3|x_0|^2+3|x_0|+1}\right)$
Let $x \in \mathbb{R}$ and assume $|x - x_0| < \delta$

$$|f(x) - f(x_0)| = |x^3 - x_0^3| = |x - x_0| \cdot |x^2 + xx_0 + x_0^2| < \delta(|x^2| + |x| \cdot |x_0| + |x_0^2|)$$
$$\leq \delta((1 + |x_0|)^2 + (|x_0| + 1)|x_0| + |x_0|^2) = \delta(3|x_0|^2 + 3|x_0| + 1) < \frac{3|x_0|^2 + 3|x_0| + 1}{3|x_0|^2 + 3|x_0| + 1}\varepsilon = \varepsilon$$

24) State what it means to be periodic, then prove that a periodic function $f: \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

f is periodic if there is a period p such that $f(x) = f(x + p)$. Weird, something is missing. I think perhaps we meant to assume f is continuous. Because this is very much not true. Consider the periodic function below that is not uniformly continuous.

$$f(x) = \vdash \begin{cases} \tan(x), \text{if } x \neq \frac{\pi n}{2} \\ 0, \text{if } x = \frac{n\pi}{2} \end{cases}$$

25) Let $f: [a,b] \to \mathbb{R}$ be continuous and one-to-one. Assume $f(a) < f(b)$ and $c \in [a,b]$.
Prove that $c < f(b)$.

This question is all messed up. In fact, as stated it's easy to come up with a counterexample such as below. Without asking for any clarification during the test, that's how to answer a messed up question. Treat it exactly as stated. Better to ask about it if you're confused and suspect something is amiss.

This is actually false. Take $f: [0,1] \to \mathbb{R}$ given by $f(x) = -5 + x$. This is continuous and one to one with $f(0) < f(1)$. Then take $c = \frac{1}{2}$. However, $\frac{1}{2} \nless f(1)$.

This is what I intended:
Let $f: [a, b] \to \mathbb{R}$ be continuous and one-to-one. Assume $f(a) < f(b)$ and $c \in (a, b)$.
Prove that $f(c) < f(b)$.

Suppose that $f(c) \geq f(b)$. Then consider the real number $m \coloneqq \frac{f(c)+f(b)}{2}$. The intermediate value theorem there is a point $x_1 \in [a, c]$ such that $f(x_1) = m$. Similarly there is a point $x_2 \in [c, b]$ such that $f(x_2) = m$. Note that because f is one to one, $f(c) \neq f(b)$, so $f(c) \neq m$. Hence $x_1, x_2 \neq c$. Thus $x_1 \neq x_2$ and so f is not one to one.

26) Prove that $f: \{0\} \to \mathbb{R}$ is continuous at 0, regardless of how f is defined.