Choose FOUR of the problems below to complete for 25 points each. Start each problem at the top of a new sheet/side of paper.

If you have time to complete all 5 problems, the problem on the last page will be counted for up to 5 bonus points.

1) Solve \( z^4 + 81 = 0 \) completely.

\[
z^4 = -81 = 81e^{\pi i} = 81e^{3\pi i} = 81e^{5\pi i} = 81e^{7\pi i}
\]

\[
z \in \left\{ 3e^{\frac{\pi i}{4}}, 3e^{\frac{3\pi i}{4}}, 3e^{\frac{5\pi i}{4}}, 3e^{\frac{7\pi i}{4}} \right\}
\]
2) Define the square $S$ in the complex plane as the square that has vertices at $0, i, -1 + i, \text{ and } -1$. Let $f(z) = z^2$. Precisely find the region, $f(S)$, the region that the square maps to via the function $f$.

Intuitively we know that this region will look like some kind of parabolic-ish region in quadrants III and IV. To determine it precisely, we’ll find the boundary equations:

$S$ is contained by the four contours:

$$C_1 = \{-t| t \in [0,1]\}$$
$$C_2 = \{it| t \in [0,1]\}$$
$$C_3 = \{-t + i| t \in [0,1]\}$$
$$C_4 = \{-1 + it| t \in [0,1]\}$$
Because $f$ is a continuous function, the region $f(S)$ will be contained by the region bounded by the $f(C_i)$’s. Let’s find these regions:

- $f(C_1) = \{t^2 | t \in [0,1]\} = \{t | t \in [0,1]\}$
- $f(C_2) = \{-t^2 | t \in [0,1]\} = \{-t | t \in [0,1]\}$
- $f(C_3) = \{(-t + i)^2 | t \in [0,1]\} = \{-1 + t^2 + i(-2t) | t \in [0,1]\}$
- $f(C_4) = \{(-1 + it)^2 | t \in [0,1]\} = \{1 - t^2 + i(-2t) | t \in [0,1]\}$

For $f(C_3)$, we can view this as a curve in the $x – y$ plane given by $x = -1 + t^2$ and $y = -2t$. This works out to:

\[
\begin{align*}
x &= -1 + \left(\frac{y}{-2}\right)^2 \\
y^2 &= 4x + 4 \\
x &\in [-1,0] \\
y &\in [-1,0]
\end{align*}
\]

Similarly for $f(C_4)$:

\[
\begin{align*}
x &= 1 - t^2, y = -2t \\
y^2 &= 4 - 4x \\
x &\in [0,1] \\
y &\in [-1,0]
\end{align*}
\]
3) Let $z_1$ and $z_2$ be complex numbers. Prove that $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$.

Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where $x_1, x_2, y_1,$ and $y_2 \in \mathbb{R}$. Then we may compute the left and right hand sides of this, noting that they are the same:

$$\overline{z_1 z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)} = x_1 x_2 - y_1 y_2 - i(x_1 y_2 + x_2 y_1)$$

$$\overline{z_1} \overline{z_2} = (\overline{x_1 + iy_1}) \cdot (\overline{x_2 + iy_2}) = (x_1 - iy_1)(x_2 - iy_2) = x_1 x_2 - y_1 y_2 - i(x_1 y_2 + x_2 y_1)$$

Also note that this can be done in exponential form as well, and it is perhaps a little easier that way.
4) Find the limit below.

$$\lim_{z \to e^{\frac{4\pi i}{3}}} \left( z - e^{\frac{4\pi i}{3}} \right) \left( \frac{z^2}{z^3 - 1} \right)$$

First note that \( (z - e^{\frac{4\pi i}{3}}) \left( \frac{z^2}{z^3 - 1} \right) \) is not continuous at \( e^{\frac{4\pi i}{3}} \) because it does not exist there; hence we cannot plug \( e^{\frac{4\pi i}{3}} \) in for \( z \). However, it is a removable discontinuity so the limit will be the same if we factor and cancel:

\[
\left( z - e^{\frac{4\pi i}{3}} \right) \left( \frac{z^2}{z^3 - 1} \right) = \left( z - e^{\frac{4\pi i}{3}} \right) \left( \frac{z^2}{(z - 1) \left( z - e^{\frac{-4\pi i}{3}} \right) \left( z - e^{\frac{4\pi i}{3}} \right)} \right) = \frac{z^2}{(z - 1) \left( z - e^{\frac{2\pi i}{3}} \right)}
\]

Hence we get:

\[
\lim_{z \to e^{\frac{4\pi i}{3}}} \left( z - e^{\frac{4\pi i}{3}} \right) \left( \frac{z^2}{z^3 - 1} \right) = \lim_{z \to e^{\frac{4\pi i}{3}}} \frac{z^2}{(z - 1) \left( z - e^{\frac{-4\pi i}{3}} \right) \left( z - e^{\frac{4\pi i}{3}} \right)} = \frac{\left( e^{\frac{4\pi i}{3}} \right)^2}{(e^{\frac{4\pi i}{3}} - 1)(e^{\frac{-4\pi i}{3}} - e^{\frac{4\pi i}{3}})}
\]
5) Let \( z = x + iy \) where \( x \) and \( y \) are real variables. Is the function \( f(z) \) given below an analytic function? State all theorems that you use.

\[
f(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)
\]

The theorem we use here is regarding the Cauchy Riemann equations. It says that if we write \( f(z) = u(x, y) + iv(x, y) \) where \( x \) and \( y \) are real variables, then the function \( f \) is analytic if and only if \( u_x, u_y, v_x, \) and \( v_y \) are continuous and satisfy the two equations below:

\[
\begin{align*}
    u_x &= v_y \\
    u_y &= -v_x
\end{align*}
\]

Indeed, they are equal as \( u_x = v_y = 6x + 2, u_y = -6y, \) and \( v_x = 6y \). Note also that these are polynomials, which are continuous.