Name ___Solutions_______________________________________Complex Analysis, Spring 2017, Test 1

Choose FOUR of the problems below to complete for 25 points each. Start each problem at the top of a new sheet/side of paper.

If you have time to complete all 5 problems, the problem on the last page will be counted for up to 5 bonus points.

1) Solve $z^4 + 81 = 0$ completely.

$$z^4 = -81 = 81e^{\pi i} = 81e^{3\pi i} = 81e^{5\pi i} = 81e^{7\pi i}$$

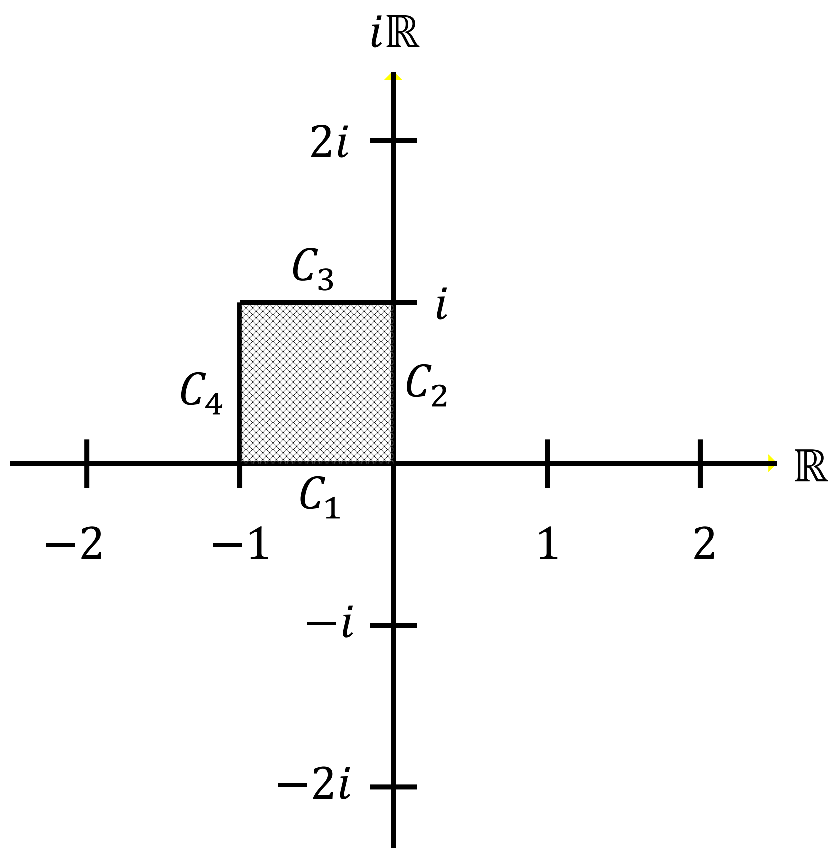
$$z \in \left\{3e^{\frac{\pi i}{4}}, 3e^{\frac{3\pi i}{4}}, 3e^{\frac{5\pi i}{4}}, 3e^{\frac{7\pi i}{4}}\right\}$$

2) Define the square $S$ in the complex plane as the square that has vertices at $0, i, -1+i$, and $-1$. Let $f(z) = z^2$. Precisely find the region, $f(S)$, the region that the square maps to via the function $f$.

Intuitively we know that this region will look like some kind of parabolic-ish region in quadrants III and IV. To determine it precisely, we'll find the boundary equations:

$S$ is contained by the four contours:

$$C_1 = \{-t | t \in [0,1]\}$$
$$C_2 = \{it | t \in [0,1]\}$$
$$C_3 = \{-t + i | t \in [0,1]\}$$
$$C_4 = \{-1 + it | t \in [0,1]\}$$



Because $f$ is a continuous function, the region $f(S)$ will be contained by the region bounded by the $f(C_i)'s$. Let's find these regions:

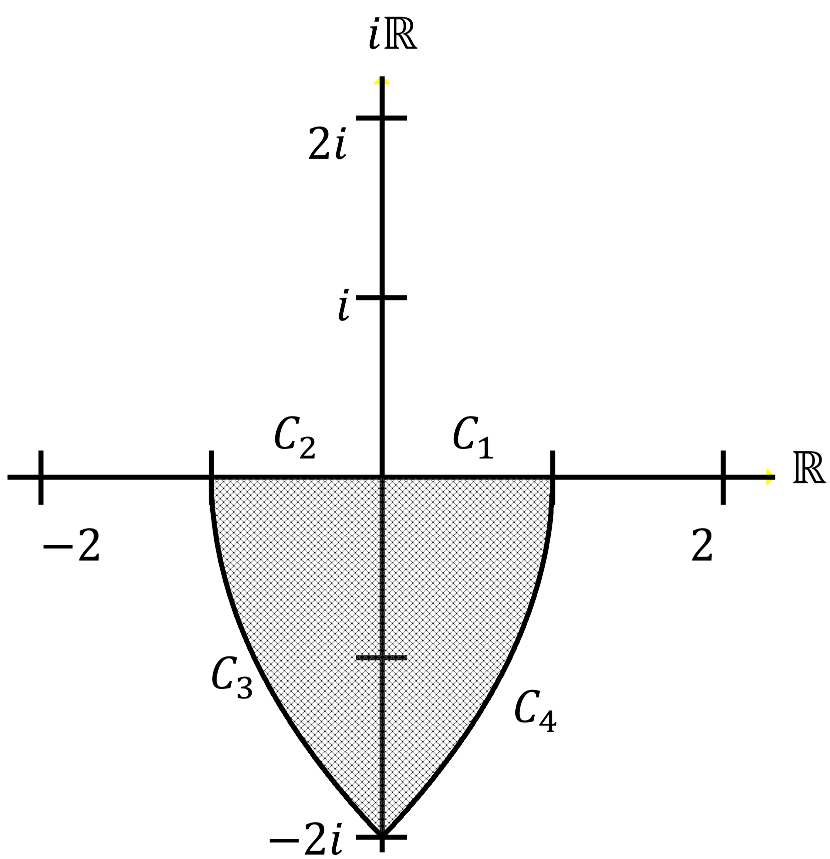
$$f(C_1) = \{t^2 | t \in [0,1]\} = \{t | t \in [0,1]\}$$
$$f(C_2) = \{-t^2 | t \in [0,1]\} = \{-t | t \in [0,1]\}$$
$$f(C_3) = \{(-t+i)^2 | t \in [0,1]\} = \{-1 + t^2 + i(-2t) | t \in [0,1]\}$$
$$f(C_4) = \{(-1+it)^2 | t \in [0,1]\} = \{1 - t^2 + i(-2t) | t \in [0,1]\}$$

For $f(C_3)$, we can view this as a curve in the $x - y$ plane given by $x = -1 + t^2$ and $y = -2t$. This works out to:

$$x = -1 + \left(\frac{y}{-2}\right)^2$$
$$y^2 = 4x + 4$$
$$x \in [-1,0]$$
$$y \in [-1,0]$$

Similarly for $f(C_4)$:

$$x = 1 - t^2, y = -2t$$
$$y^2 = 4 - 4x$$
$$x \in [0,1]$$
$$y \in [-1,0]$$



3) Let $z_1$ and $z_2$ be complex numbers. Prove that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where $x_1, x_2, y_1,$ and $y_2 \in \mathbb{R}$. Then we may compute the left and right hand sides of this, noting that they are the same:

$$\overline{z_1 z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)} = x_1 x_2 - y_1 y_2 - i(x_1 y_2 + x_2 y_1)$$

$$\overline{z_1}\,\overline{z_2} = \left(\overline{x_1 + iy_1}\right) \cdot \left(\overline{x_2 + iy_2}\right) = (x_1 - iy_1)(x_2 - iy_2) = x_1 x_2 - y_1 y_2 - i(x_1 y_2 + x_2 y_1)$$

Also note that this can be done in exponential form as well, and it is perhaps a little easier that way.

Type equation here.

4) Find the limit below.

$$\lim_{z \to e^{\frac{4\pi}{3}i}} \left(z - e^{\frac{4\pi}{3}i}\right)\left(\frac{z^2}{z^3 - 1}\right)$$

First note that $\left(z - e^{\frac{4\pi}{3}i}\right)\left(\frac{z^2}{z^3-1}\right)$ is not continuous at $e^{\frac{4\pi i}{3}}$ because it does not exist there; hence we cannot plug $e^{\frac{4\pi i}{3}}$ in for $z$. However, it is a removable discontinuity so the limit will be the same if we factor and cancel:

$$\left(z - e^{\frac{4\pi}{3}i}\right)\left(\frac{z^2}{z^3 - 1}\right) = \left(z - e^{\frac{4\pi}{3}i}\right)\left(\frac{z^2}{(z-1)\left(z - e^{\frac{2\pi i}{3}}\right)\left(z - e^{\frac{4\pi i}{3}}\right)}\right) = \frac{z^2}{(z-1)\left(z - e^{\frac{2\pi i}{3}}\right)}$$

Hence we get:

$$\lim_{z \to e^{\frac{4\pi}{3}i}} \left(z - e^{\frac{4\pi}{3}i}\right)\left(\frac{z^2}{z^3 - 1}\right) = \lim_{z \to e^{\frac{4\pi}{3}i}} \frac{z^2}{(z-1)\left(z - e^{\frac{2\pi i}{3}}\right)} = \frac{\left(e^{\frac{4\pi i}{3}}\right)^2}{\left(e^{\frac{4\pi i}{3}} - 1\right)\left(e^{\frac{4\pi i}{3}} - e^{\frac{2\pi i}{3}}\right)}$$

5) Let $z = x + iy$ where $x$ and $y$ are real variables. Is the function $f(z)$ given below an analytic function? State all theorems that you use.

$$f(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$$

The theorem we use here is regarding the Cauchy Riemann equations. It says that if we write $f(z) = u(x, y) + iv(x, y)$ where $x$ and $y$ are real variables, then the function $f$ is analytic if and only if $u_x, u_y, v_x,$ and $v_y$ are continuous and satisfy the two equations below:

$$u_x = v_y$$
$$u_y = -v_x$$

Indeed, they are equal as $u_x = v_y = 6x + 2$, $u_y = -6y$, and $v_x = 6y$. Note also that these are polynomials, which are continuous.