Name ____Complex Analysis, Spring 2017, Test 2

Choose FOUR of the problems below to complete for 25 points each. Start each problem at the top of a new sheet/side of paper.

If you have time to complete all 5 problems, note on *this* page which one you want to be a bonus. It will be counted for up to 15 bonus points.

1) Find all the values of $\left|(-i)^{-i}\right|$.

$$\left|(-i)^{-i}\right| = \left|e^{\ln\left((-i)^{-i}\right)}\right|$$
$$= \left|e^{-i\ln(-i)}\right|$$
$$= \left|e^{-i(\ln|-i|+i\arg(-i)+2\pi ki)}\right|$$
$$= \left|e^{-i\left(\ln|1|+\frac{3\pi i}{2}+2\pi ki\right)}\right|$$
$$= \left|e^{-i\left(\frac{3\pi i}{2}+2\pi ki\right)}\right|$$
$$= \left|e^{\frac{3\pi}{2}+2\pi k}\right|$$
$$= e^{\frac{3\pi}{2}}e^{2\pi k}$$

2) This question has three parts. Find all branch points for $f(z) = \ln(z - z^2)$. Define the principle branch of $f(z)$. Illustrate an appropriate branch cut.

$g(z) = \ln(z)$ has branch points when $z = 0$ or $z = \infty$. So let's investigate when $z - z^2$ is 0 or ∞. This is when $z = 0, z = 1$, or $z = \infty$. Hence these are the branch points.

For a branch cut, we need to connect all of these to prevent circling about any of them. The positive x-axis connecting $z = 0$ and $z = \infty$ via the point $z = 1$ is the simplest branch cut. (It takes some more work to justify this precisely, but that was not necessary)

The principle branch is the branch in which $k = 0$; that is, in which $\ln(z - z^2) = \ln|z - z^2| + i \arg(z - z^2)$.

3) Find $\int_C \bar{z}\, dz$ where C is given by the cubic $x = y^3$ from $(0,0)$ to $(8,2)$.

Method 1: Parametrize it using $y = t$ and $x = t^3$. Then $z = t^3 + it$ and so $dz = (3t^2 + i)dt$.

$$\int_C \bar{z} dz = \int_0^2 \overline{t^3 + it}\,(3t^2 + i)dt = \int_0^2 (t^3 - it)(3t^2 + i)dt = 34 - 8i$$

Method 2: Keep it in x's and y's using $x = y^3$ and $y = x^{\frac{1}{3}}$:

$$\int_C \bar{z} dz = \int_C \overline{x + iy}(dx + idy) = \int_C (x - iy)(dx + idy) = \int_C (x - iy)dx + i\int_C (x - iy)dy$$

$$= \int_0^8 \left(x - ix^{\frac{1}{3}}\right) dx + i\int_0^2 (y^3 - iy)dy = 34 - 8i$$

4) Find $\int_C \frac{dz}{z^2-9}$ where C is the circle given by $|z-2|=4$. Be sure that your work justifies your answer.

Note that C includes the singularity at $z=3$, but not the one at $z=-3$. We could use Cauchy's Integral Formula, but I'm going to use that for problem #5. Hence we'll use Cauchy's theorem for this one and reduce the problem to a simpler integral. (See picture if I get around to making it)

$$\int_C \frac{dz}{z^2-9} = \frac{1}{6}\int_C \frac{1}{z-3}dz - \frac{1}{6}\int_C \frac{1}{z+3}dz = 0 - \frac{1}{6}\int_C \frac{1}{z+3}dz = -\frac{1}{6}\int_0^{\pi i} \frac{1}{e^{i\theta}} ie^{i\theta}\,d\theta = -\frac{2\pi i}{6} = -\frac{\pi i}{3}$$

5) Find $\int_C \frac{\cos(\pi z)}{z^2-1} dz$ where C is the rectangle with corners at $-i, 2-i, 2+i$, and i. Be sure that your work justifies your answer.

Here we note that the contour C includes the singularity $z = 1$ but not $z = -1$. Hence we use Cauchy's integral formula, identifying' $f(z) = \frac{\cos(\pi z)}{z+1}$, and then obtain:

$$\int_C \frac{\cos(\pi z)}{z^2 - 1} dz = f(1) \cdot 2\pi i = \frac{\cos(\pi)}{2} \cdot 2\pi i = -\frac{1}{2} \cdot 2\pi i = -\pi i$$