Divisibility of an Eigenform by another Eigenform

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September 25, 2011
Modular forms for level $\Gamma = SL_2(\mathbb{Z})$

Remark

A modular form in this talk will mean a modular form in level 1 of positive even weight $k \geq 4$
Example

The weight $k$ Eisenstein series is a modular form, given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

Example

The weight 12 cusp form is $\Delta(z)$ given by

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$
Fact

Let $\mathcal{E}_k$ be the space of weight $k$ Eisenstein series, and $S_k$ the space of weight $k$ cusp forms. Let $M_k$ be the space of all weight $k$ modular forms. Then

$$M_k = \mathcal{E}_k \oplus S_k$$
Hecke Operators

Definition

The Hecke operator $T_{n,k}$ is a linear operator $M_k \rightarrow M_k$ given by

$$(T_{n,k}(f))(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f \left( \frac{nz + bd}{d^2} \right).$$

Definition

The Hecke polynomial $T_{n,k}(x)$ is the characteristic polynomial of the operator $T_{n,k}$ on $S_k$.

Remark

So if $\lambda_i$ are the eigenvalues of $T_{n,k}$, then $T_{n,k}(x) = \prod (x - \lambda_i)$.
**Definition**

A modular form \( f \in M_k \) is said to be an eigenform if it is an eigenvector for all the Hecke operators \( \{ T_{n,k} \}_{n \in \mathbb{N}} \).

**Fact**

\( M_k \) has a basis of eigenforms.

**Example**

\( M_{12} \) is generated by \( E_{12} \) and \( \Delta(z) \).

**Example**

\( S_k \) has dimension 1 for \( k \in \{ 12, 16, 18, 20, 22, 26 \} \). In this case let \( \Delta_k(z) \) be the unique normalized cusp form in \( S_k \). In particular \( \Delta_{12}(z) = \Delta(z) \).
More on Modular Forms

**Fact**

*Hecke operators preserve Eisenstein series (resp. Cusp forms)*

**Remark**

*In particular, because \( \dim(\mathcal{E}_k) = 1 \), every \( f \in \mathcal{E}_k \) is an eigenform. (resp. if \( \dim(\mathcal{S}_k) = 1 \) then every \( f \in \mathcal{S}_k \) is an eigenform)*
The Question

Given an eigenform $h$, is $h$ divisible by another eigenform $f$?

There are three cases:

- $h$ is cuspidal, $f$ is an Eisenstein series.
- $h$ is cuspidal, $f$ is cuspidal.
- $h$ and $f$ are both Eisenstein series.
Products of eigenforms

**Example**

$E_{10}$ is an eigenform, and $E_{10} = E_4 E_6$.

**Example**

$\Delta$ divides every cuspidal modular form.

**Remark**

*if $fg$ is an eigenform, then one of $f, g$ must not be a cusp form.*
Example

Let \( r > 26, \ r \equiv 4 \mod 12 \) then \( S_r = E_4 \Delta M_{r-16} \) so that every \( h \in S_r \) factors as \( h = E_4 \Delta g \) for some \( g \in M_{r-16} \).

Example

Consider \( h = E_{16} \Delta - \frac{14903892}{3617} E_4 \Delta^2 - 108 \sqrt{18209} E_4 \Delta^2 \), which is an eigenform in \( S_{28} \). Then one factorization of \( h \) is:

\[
h = E_4 \Delta \left( E_{12} - \frac{3075516}{691} \Delta - 108 \sqrt{18209} \Delta \right)
\]
Theorem

The product of two eigenforms is an eigenform only in the following cases:

- \( E_4^2 = E_8 \)
- \( E_4 E_6 = E_{10} \)
- \( E_4 \Delta_{12} = \Delta_{16} \)
- \( E_4 \Delta_{16} = E_8 \Delta_{12} = \Delta_{20} \)
- \( E_4 \Delta_{18} = E_6 \Delta_{16} = E_{10} \Delta_{12} = \Delta_{22} \)
- \( E_4 \Delta_{22} = E_6 \Delta_{20} = E_8 \Delta_{18} = E_{10} \Delta_{12} = E_{14} \Delta_{12} = \Delta_{26} \)
- \( E_6 E_8 = E_4 E_{10} = E_{14} \)
- \( E_6 \Delta_{12} = \Delta_{18} \)
Generalizations

- 2007 - Emmons and Lanphier generalized Ghate and Duke’s work to an arbitrary number of eigenforms
- 2004 - Lanphier and Takloo-Bighash generalized Ghate and Duke’s work to Rankin-Cohen bracket operators
- 2010 - Under the guidance of James and Xue, Trentacoste and myself generalized Ghate and Duke’s work to nearly holomorphic modular forms.
Notations and stuff

- $h = \Box g$, $h, \Box$ eigenforms, $g$ a modular form.
- $h \in M_{\text{wt}(h)}$ is the “big” form.
- $g \in M_{\text{wt}(g)}$ is the “small” form.
- $S_{\text{wt}(h)} = \langle h_1, h_2, \ldots, h_l \rangle$ is an eigenform basis.
- $S_{\text{wt}(g)} = \langle g_1, \ldots, g_k \rangle$ is an eigenform basis.
- $T_n := T_{n,\text{wt}(h)}$
Main Result 1

**Theorem (B., James, Xue)**

If for some \( n \), \( T_n(x) \) is irreducible over \( \mathbb{Q} \), then a cuspidal eigenform \( h \) can be factored as \( E_s g = h \), with \( g \) a modular form if and only if \( \dim(M_{wt(h)}) = \dim(M_{wt(g)}) \). These are precisely the cases below.

\[
\begin{align*}
s &= 4, \quad wt(h) \equiv 4, 8, 10, 14 \pmod{12} \\
s &= 6, \quad wt(h) \equiv 6, 10, 14 \pmod{12} \\
s &= 8, \quad wt(h) \equiv 8, 14 \pmod{12} \\
s &= 10, \quad wt(h) \equiv 10, 14 \pmod{12} \\
s &= 14, \quad wt(h) \equiv 14 \pmod{12}
\end{align*}
\]
Main Result 2

Theorem (B., James, Xue)

If for some $n$, $T_n(x)$ is irreducible over every $\mathbb{F}$ of degree less than $\dim(S_{\text{wt}(h)})$, then a cuspidal eigenform $h \in S_{\text{wt}(h)}$ can be factored as $h = fg$ where $g$ a modular form if and only if $\dim(S_{\text{wt}(h)}) = \dim(M_{\text{wt}(g)})$. These are precisely the cases below.

- $\text{wt}(f) = 12, \text{wt}(h) \equiv 4, 6, 8, 10, 12, 14 \mod (12)$
- $\text{wt}(f) = 16, \text{wt}(h) \equiv 4, 8, 10, 14 \mod (12)$
- $\text{wt}(f) = 18, \text{wt}(h) \equiv 6, 10, 14 \mod (12)$
- $\text{wt}(f) = 20, \text{wt}(h) \equiv 8, 14 \mod (12)$
- $\text{wt}(f) = 22, \text{wt}(h) \equiv 10, 14 \mod (12)$
- $\text{wt}(f) = 26, \text{wt}(h) \equiv 14 \mod (12)$
Main Result 3

**Definition**

\[ \varphi_{wt}(h) := \prod (x - j_i), \]  
where the product runs over all the \( j \)-zeros of \( E_{wt}(h) \) except for 0 and 1728. This function satisfies \[ \frac{E_{wt}(h)}{E_4^a E_6^b \Delta^c} = \varphi_{wt}(h)(j). \]

**Theorem (B., James, Xue)**

If \( \varphi_{wt}(h) := \prod (x - j_i) \) is irreducible over \( \mathbb{Q} \), then \( E_s \) divides \( E_{wt}(h) \) when and only when \( \dim(M_{wt(h)}) = \dim(M_{wt(g)}) \). These are precisely the cases below.

- \( s = 4, \; wt(h) \equiv 4, 8, 12, 14 \mod (12) \)
- \( s = 6, \; wt(h) \equiv 6, 10, 14 \mod (12) \)
- \( s = 8, \; wt(h) \equiv 8, 14 \mod (12) \)
- \( s = 10, \; wt(h) \equiv 10, 14 \mod (12) \)
- \( s = 14, \; wt(h) \equiv 14 \mod (12) \)
Rational Subspace Lemma

**Definition**
A subspace of $S \subseteq S_m$ is said to be $\mathbb{F}$-rational if it is fixed under the Galois action: $\sigma(S) = S$ for all $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{F})$.

**Lemma**
*If a proper $\mathbb{F}$-rational subspace $S$ of $S_m$ contains an eigenform, then $T_n(x)$ is reducible over $\mathbb{F}$ for all $n$.***

**Corollary**
*If for some $n$, $T_n(x)$ is irreducible over all fields $\mathbb{F}$ of degree less than $\dim(S_{\omega(h)})$, then no proper $\mathbb{F}$-rational subspace of $S_m$ can contain an eigenform.*
Proof of the rational subspace lemma

Let $S \subset S_m$ be a proper $\mathbb{F}$-rational subspace of degree less than $\dim(S_{wt(h)})$ containing an eigenform $h \in S$. Then define

$$R := \langle \sigma(h) \mid \sigma \in \text{Gal}(\mathbb{C}/\mathbb{F}) \rangle_{\mathbb{C}} \leq S$$

which is an $\mathbb{F}$-rational space.

Then $S_m = R \oplus R^\perp$, both of which are stable under Hecke operators because they have eigenform bases. Thus $T_n(x) = T_n|_R(x) \cdot T_n|_{R^\perp}(x)$. Next note that each of $T_n|_R(x)$ has coefficients in $\mathbb{F}$ because its coefficients are in fact the symmetric polynomials on the $\mathbb{F}$ conjugates of an eigenvalue $\lambda$:

$$T_n|_R(x) = \prod_{\sigma} (x - \sigma(\lambda))$$

where the product is over a finite set of $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{F})$ such that $\sigma(h)$ are distinct. Hence $T_n|_{R^\perp}$ also has $\mathbb{F}$-rational coefficients, and so we have that $T_n(x)$ is reducible over $\mathbb{F}$. 
Proof: Main Result 1 \((h = E_s g)\)

**Proof.**

If we are not in the cases stated in the conjecture, then \(\dim(M_{wt(h)}) > \dim(M_{wt(g)})\), that is, the dimension of the space that \(h\) lives in is strictly larger than the space that \(g\) lives in. Then \(E_s S_{wt(g)} = \langle E_s g_1, \ldots, E_s g_k \rangle\) is a proper \(\mathbb{Q}\)-rational subspace of \(S_{wt(h)}\), and thus \(T_n(x)\) is reducible over \(\mathbb{Q}\) for all \(n\).

If \(\dim(M_{wt(h)}) = \dim(M_{wt(g)})\) then we can construct an example using linear algebra.

**Remark**

If \(\dim(S_{wt(h)}) = 1\), then we are in the case of Ghate and Duke and in fact \(g\) turns out to be an eigenform.
Next we will present an equivalent theorem regarding $L$-values.

**Definition**

Let $g = \sum a_n q^n$ and $h = \sum b_n q^n$ be cusp forms.

\[
L(g, h) := L(g \times h, \text{wt}(h) - 1) = \sum_{i=1}^{\infty} \frac{a_i \overline{b_i}}{i^{\text{wt}(h) - 1}}.
\]
Another theorem

Theorem

If $T_n(x)$ is irreducible over $\mathbb{Q}$ for some $n$, then for $l > k$ the vectors of $L$-values given below are linearly independent over $\mathbb{C}$, and when $l = k$ there is a single dependency relation.

\[
\left\{ \begin{bmatrix} L(g_1, h_1) \\ \vdots \\ L(g_1, h_{i-1}) \\ L(g_1, h_{i+1}) \\ \vdots \\ L(g_1, h_l) \end{bmatrix} \right\}, \ldots, \left\{ \begin{bmatrix} L(g_k, h_1) \\ \vdots \\ L(g_k, h_{i-1}) \\ L(g_k, h_{i+1}) \\ \vdots \\ L(g_k, h_l) \end{bmatrix} \right\}
\] (1)
The Rankin-Selberg Method

Theorem

Let \( g = \sum a_n q^n \) and \( h = \sum b_n q^n \) be cusp forms.

\[
\langle E_s \cdot g, h \rangle = (4\pi)^{-(s + \operatorname{wt}(g) - 1)} \Gamma(s + \operatorname{wt}(g) - 1) \sum_{n \geq 1} \frac{a_n \overline{b_n}}{n^{s + \operatorname{wt}(g) - 1}}
\]
Proof

Suppose that $T_n(x)$ is irreducible over $\mathbb{Q}$ for some $n$. For convenience write

$$L := \begin{bmatrix}
L(g_1, h_2) & \cdots & L(g_k, h_2) \\
\vdots & \ddots & \vdots \\
L(g_1, h_l) & \cdots & L(g_k, h_l)
\end{bmatrix}$$

Suppose there is a solution $[c_1, \ldots, c_k]^T$ to the matrix equation $L \vec{x} = \vec{0}$. We must show that $[c_1, \ldots, c_k]^T = \vec{0}$. We have:

$$c_1 L(g_1, h_i) + \cdots + c_k L(g_k, h_i) = 0 \quad i = 2, \ldots, l$$

By using the Rankin-Selberg method and denoting $g := c_1 E_s g_1 + \cdots c_k E_s g_k$ we have $\langle g, h_i \rangle = 0$ for $i = 2, \ldots, l$. Hence $g$ is orthogonal to each of $h_2, h_3, \ldots, h_l$. Thus by linear algebra $E_s g = ch_1$. By the previous theorem, this cannot occur nontrivially, and so $g = 0$ and $c = 0$. In particular, $c_1 = \cdots = c_k = 0$. 

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Remark

We used several times that for some $n$, $T_n(x)$ is irreducible over every $\mathbb{F}$ of degree less than $\dim(S_{wt(h)})$. We have reason to believe that this is usually the case.

Conjecture (Maeda, 1997)

The Hecke algebra over $\mathbb{Q}$ of $S_m(SL_2(\mathbb{Z}))$ is simple (that is, a single number field) whose Galois closure over $\mathbb{Q}$ has Galois group isomorphic to a symmetric group $S_l$ (with $l = \dim(S_m(SL_2(\mathbb{Z})))$).

Remark

An implication of Maeda’s conjecture that we will use is that $T_n(x)$ has full Galois group. In particular this implies that $T_n(x)$ is irreducible over every $\mathbb{F}$ of degree less than degree of $T_n(x)$.
Maeda’s Conjecture

Remark

This conjecture was presented in a paper by Hide and Maeda in 1997 and was verified for weights less less than 469. Later Farmer and James verified it up to weight 2000, up to to weight 3000 by Kleinerman. $T_n(x)$ was found irreducible up to weight 4096 by Ghitza. We showed $\varphi_k$ to be irreducible up to weight 1500. In particular, for these weights our theorems are true without condition.
The End