Divisibility of an Eigenform by another Eigenform

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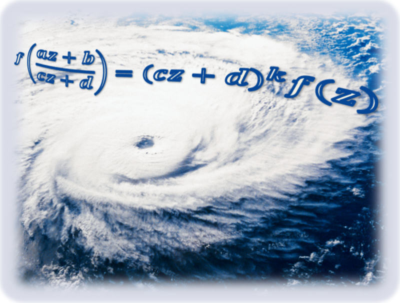
Modular forms for level $\Gamma = SL_2(\mathbb{Z})$

Remark

A modular form in this talk will mean a modular form in level 1 *of positive even weight* $k \geq 4$



Modular Forms

Example

The weight k Eisenstein series is a modular form, given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

Example

The weight 12 cusp form is $\Delta(z)$ given by

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

Modular Forms

Fact

Let $\mathcal{E}_k$ be the space of weight k Eisenstein series, and S_k the space of weight k cusp forms. Let M_k be the space of all weight k modular forms. Then

$$M_k = \mathcal{E}_k \oplus S_k$$

Hecke Operators

Definition

The Hecke operator $T_{n,k}$ is a linear operator $M_k \to M_k$ given by

$$\left(T_{n,k}\left(f\right)\right)(z) = n^{k-1}\sum_{d|n} d^{-k}\sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right).$$

Definition

The Hecke polynomial $T_{n,k}(x)$ is the characteristic polynoimal of the operator $T_{n,k}$ on S_k.

Remark

So if λ_i are the eigenvalues of $T_{n,k}$, then $T_{n,k}(x) = \prod(x-\lambda_i)$

More on Modular Forms

Definition

A modular form $f \in M_k$ is said to be an eigenform if it is an eigenvector for all the Hecke operators $\{T_{n,k}\}_{n\in\mathbb{N}}$.

Fact

M_k has a basis of eigenforms.

Example

M_{12} is generated by E_{12} and $\Delta(z)$.

Example

S_k has dimension 1 for $k \in \{12, 16, 18, 20, 22, 26\}$. In this case let $\Delta_k(z)$ be the unique normalized cusp form in S_k. In particular $\Delta_{12}(z) = \Delta(z)$.

More on Modular Forms

Fact

Hecke operators preserve Eisenstein series (resp. Cusp forms)

Remark

In particular, because $\dim(\mathcal{E}_k) = 1$*, every* $f \in \mathcal{E}_k$ *is an eigenform.*
(resp. if $\dim(S_k) = 1$ *then every* $f \in S_k$ *is an eigenform)*

The Question

Question

Given an eigenform h, is h divisible by another eigenform f?
There are three cases:

- *h is cuspidal, f is an Eisenstein series.*
- *h is cuspidal, f is cuspidal.*
- *h and f are both Eisenstein series.*

Products of eigenforms

Example

E_{10} is an eigenform, and $E_{10} = E_4 E_6$.

Example

Δ divides every cuspidal modular form.

Remark

if fg is an eigenform, then one of f, g must not be a cusp form.

Example

Let $r > 26$, $r \equiv 4 \bmod 12$ then $S_r = E_4 \Delta M_{r-16}$ so that every $h \in S_r$ factors as $h = E_4 \Delta g$ for some $g \in M_{r-16}$.

Example

Consider $h = E_{16}\Delta - \frac{14903892}{3617} E_4 \Delta^2 - 108\sqrt{18209} E_4 \Delta^2$, which is an eigenform in S_{28}. Then one factorization of h is:

$$h = E_4 \Delta \left(E_{12} - \frac{3075516}{691} \Delta - 108\sqrt{18209} \Delta \right)$$

Results of Ghate and Duke

Theorem

The product of two eigenforms is an eigenform only in the following cases:

- $E_4^2 = E_8$
- $E_4 E_6 = E_{10}$
- $E_4 \Delta_{12} = \Delta_{16}$
- $E_4 \Delta_{16} = E_8 \Delta_{12} = \Delta_{20}$
- $E_4 \Delta_{18} = E_6 \Delta_{16} = E_{10} \Delta_{12} = \Delta_{22}$,
- $E_4 \Delta_{22} = E_6 \Delta_{20} = E_8 \Delta_{18} = E_{10} \Delta_{12} = E_{14} \Delta_{12} = \Delta_{26}$.
- $E_6 E_8 = E_4 E_{10} = E_{14}$
- $E_6 \Delta_{12} = \Delta_{18}$

Generalizations

- 2007 - Emmons and Lanphier generalized Ghate and Duke's work to an arbitrary number of eigenforms
- 2004 - Lanphier and Takloo-Bighash generalized Ghate and Duke's work to Rankin-Cohen bracket operators
- 2010 - Under the guidance of James and Xue, Trentacoste and myself generalized Ghate and Duke's work to nearly holomorphic modular forms.

Notations and stuff

- $h = \Box g$, $h, \Box$ eigenforms, g a modular form.
- $h \in M_{wt(h)}$ is the "big" form.
- $g \in M_{wt(g)}$ is the "small" form.
- $S_{wt(h)} = \langle h_1, h_2, ..., h_l \rangle$ is an eigenform basis.
- $S_{wt(g)} = \langle g_1, ..., g_k \rangle$ is an eigenform basis.
- $T_n := T_{n,wt(h)}$

Main Result 1

Theorem (B., James, Xue)

If for some n, $T_n(x)$ is irreducible over $\mathbb{Q}$, then a cuspidal eigenform h can be factored as $E_s g = h$, with g a modular form if and only if $\dim(M_{wt(h)}) = \dim(M_{wt(g)})$. These are precisely the cases below.

$$s = 4, wt(h) \equiv 4, 8, 10, 14 \mod (12)$$

$$s = 6, wt(h) \equiv 6, 10, 14 \mod (12)$$

$$s = 8, wt(h) \equiv 8, 14 \mod (12)$$

$$s = 10, wt(h) \equiv 10, 14 \mod (12)$$

$$s = 14, wt(h) \equiv 14 \mod (12)$$

Main Result 2

Theorem (B., James, Xue)

If for some n, $T_n(x)$ is irreducible over every $\mathbb{F}$ of degree less than $\dim(S_{wt(h)})$, then a cuspidal eigenform $h \in S_{wt(h)}$ can be factored as $h = fg$ where g a modular form if and only if $\dim(S_{wt(h)}) = \dim(M_{wt(g)})$. These are precisely the cases below.

$$wt(f) = 12, wt(h) \equiv 4, 6, 8, 10, 12, 14 \mod (12)$$
$$wt(f) = 16, wt(h) \equiv 4, 8, 10, 14 \mod (12)$$
$$wt(f) = 18, wt(h) \equiv 6, 10, 14 \mod (12)$$
$$wt(f) = 20, wt(h) \equiv 8, 14 \mod (12)$$
$$wt(f) = 22, wt(h) \equiv 10, 14 \mod (12)$$
$$wt(f) = 26, wt(h) \equiv 14 \mod (12)$$

Main Result 3

Definition

$\varphi_{wt(h)} := \prod(x - j_i)$, where the product runs over all the j-zeros of $E_{wt(h)}$ except for 0 and 1728. This function satisfies $\frac{E_{wt(h)}}{E_4^a E_6^b \Delta^c} = \varphi_{wt(h)}(j)$.

Theorem (B., James, Xue)

If $\varphi_{wt(h)} := \prod(x - j_i)$ is irreducible over $\mathbb{Q}$, then E_s divides $E_{wt(h)}$ when and only when $\dim(M_{wt(h)}) = \dim(M_{wt(g)})$. These are precisely the cases below.

$$s = 4, wt(h) \equiv 4, 8, 12, 14 \mod (12)$$
$$s = 6, wt(h) \equiv 6, 10, 14 \mod (12)$$
$$s = 8, wt(h) \equiv 8, 14 \mod (12)$$
$$s = 10, wt(h) \equiv 10, 14 \mod (12)$$
$$s = 14, wt(h) \equiv 14 \mod (12)$$

Rational Subspace Lemma

Definition

A subspace of $S \subseteq S_m$ is said to be $\mathbb{F}$-rational if it is fixed under the Galois action: $\sigma(S) = S$ for all $\sigma \in Gal(\mathbb{C}/\mathbb{F})$

Lemma

If a proper $\mathbb{F}$-rational subspace S of S_m contains an eigenform, then $T_n(x)$ is reducible over $\mathbb{F}$ for all n.

Corollary

If for some n, $T_n(x)$ is irreducible over all fields $\mathbb{F}$ of degree less than $\dim(S_{wt(h)})$*, then no proper $\mathbb{F}$-rational subspace of S_m can contain an eigenform.*

Proof of the rational subspace lemma

Let $S \subset S_m$ be a proper $\mathbb{F}$-rational subspace of degree less than $\dim(S_{wt(h)})$ containing an eigenform $h \in S$. Then define

$$R := \langle \sigma(h) | \sigma \in Gal(\mathbb{C}/\mathbb{F}) \rangle_{\mathbb{C}} \leq S$$

which is an $\mathbb{F}$-rational space.
Then $S_m = R \oplus R^{\perp}$, both of which are stable under Hecke operators because they have eigenform bases. Thus $T_n(x) = T_n|_R(x) \cdot T_n|_{R^{\perp}}(x)$. Next note that each of $T_n|_R(x)$ has coefficients in $\mathbb{F}$ because its coefficients are in fact the symmetric polynomials on the $\mathbb{F}$ conjugates of an eigenvalue λ:

$$T_n|_R(x) = \prod_{\sigma} (x - \sigma(\lambda))$$

where the product is over a finite set of $\sigma \in Gal(\mathbb{C}/\mathbb{F})$ such that $\sigma(h)$ are distinct. Hence $T_n|_{R^{\perp}}$ also has $\mathbb{F}$-rational coefficients, and so we have that $T_n(x)$ is reducible over $\mathbb{F}$.

Proof: Main Result 1 ($h = E_s g$)

Proof.

If we are not in the cases stated in the conjecture, then $\dim(M_{wt(h)}) > \dim(M_{wt(g)})$, that is, the dimension of the space that h lives in is strictly larger than the space that g lives in. Then $E_s S_{wt(g)} = \langle E_s g_1, \ldots, E_s g_k \rangle$ is a proper $\mathbb{Q}$-rational subspace of $S_{wt(h)}$, and thus $T_n(x)$ is reducible over $\mathbb{Q}$ for all n.
If $\dim(M_{wt(h)}) = \dim(M_{wt(g)})$ then we can construct an example using linear algebra. □

Remark

If $\dim(S_{wt(h)}) = 1$, *then we are in the case of Ghate and Duke and in fact* g *turns out to be an eigenform.*

L-values

Next we will present an equivalent theorem regarding L-values.

Definition

Let $g = \sum a_n q^n$ and $h = \sum b_n q^n$ be cusp forms.

$$L(g,h) := L(g \times h, wt(h) - 1) = \sum_{i=1}^{\infty} \frac{a_i \overline{b_i}}{i^{wt(h)-1}}.$$

Another theorem

Theorem

If $T_n(x)$ is irreducible over $\mathbb{Q}$ for some n, then for $l > k$ The vectors of L values given below are linearly independent over $\mathbb{C}$, and when $l = k$ there is a single dependency relation.

$$\left\{ \begin{bmatrix} L(g_1, h_1) \\ \vdots \\ L(g_1, h_{i-1}) \\ L(g_1, h_{i+1}) \\ \vdots \\ L(g_1, h_l) \end{bmatrix}, \ldots, \begin{bmatrix} L(g_k, h_1) \\ \vdots \\ L(g_k, h_{i-1}) \\ L(g_k, h_{i+1}) \\ \vdots \\ L(g_k, h_l) \end{bmatrix} \right\} \tag{1}$$

The Rankin-Selberg Method

Theorem

Let $g = \sum a_n q^n$ *and* $h = \sum b_n q^n$ *be cusp forms.*
$\langle E_s \cdot g, h \rangle = (4\pi)^{-(s+wt(g)-1)} \Gamma(s + wt(g) - 1) \sum_{n \geq 1} \frac{a_b \overline{b_n}}{n^{s+wt(g)-1}}$

Proof

Suppose that $T_n(x)$ is irreducible over $\mathbb{Q}$ for some n. For convenience write

$$L := \begin{bmatrix} L(g_1, h_2) & \cdots & L(g_k, h_2) \\ \vdots & \ddots & \vdots \\ L(g_1, h_l) & \cdots & L(g_k, h_l) \end{bmatrix}$$

Suppose there is a solution $[c_1, ..., c_k]^T$ to the matrix equation $L\overrightarrow{x} = \overrightarrow{0}$. We must show that $[c_1, ..., c_k]^T = \overrightarrow{0}$. We have:

$$c_1 L(g_1, h_i) + \cdots + c_k L(g_k, h_i) = 0 \qquad i = 2, ..., l$$

By using the Rankin-Selberg method and denoting $g := c_1 E_s g_1 + \cdots c_k E_s g_k$ we have $\langle g, h_i \rangle = 0$ for $i = 2, ..., l$. Hence g is orthogonal to each of $h_2, h_3, ..., h_l$. Thus by linear algebra $E_s g = c h_1$. By the previous theorem, this cannot occur nontrivially, and so $g = 0$ and $c = 0$. In particular, $c_1 = \cdots = c_k = 0$.

Remark

We used several times that for some n, $T_n(x)$ is irreducible over every $\mathbb{F}$ of degree less than $\dim(S_{wt(h)})$. We have reason to believe that this is usually the case.

Conjecture (Maeda, 1997)

The Hecke algebra over $\mathbb{Q}$ of $S_m(SL_2(\mathbb{Z}))$ is simple (that is, a single number field) whose Galois closure over $\mathbb{Q}$ has Galois group isomorphic to a symmetric group $\mathcal{S}_l$ (with $l = \dim(S_m(SL_2(\mathbb{Z})))$).

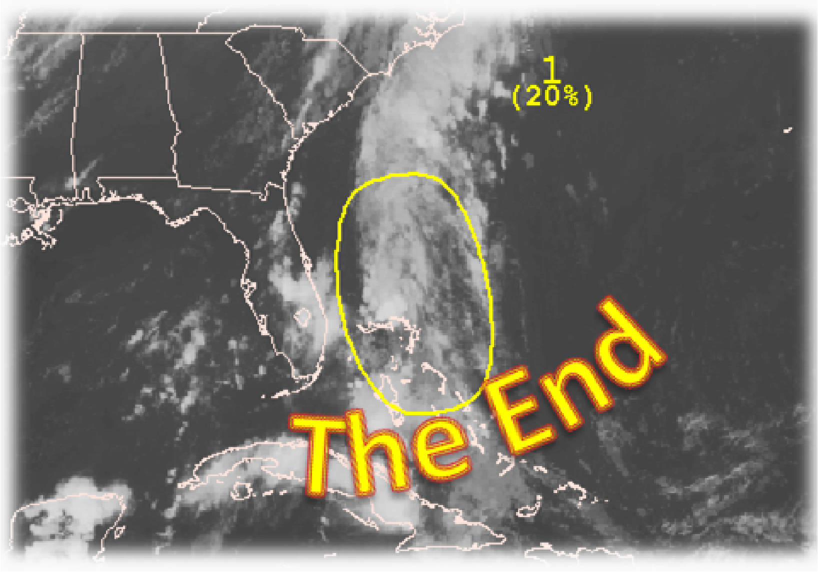
Remark

An implication of Maeda's conjecture that we will use is that $T_n(x)$ has full Galois group. In particular this implies that $T_n(x)$ is irreducible over every $\mathbb{F}$ of degree less than degree of $T_n(x)$

Maeda's Conjecture

Remark

This conjecture was presented in a paper by Hide and Maeda in 1997 and was verified for weights less less than 469. Later Farmer and James verified it up to weight 2000, up to to weight 3000 by Kleinerman. $T_n(x)$ was found irreducible up to weight 4096 by Ghitza. We showed φ_k to be irreducible up to weight 1500. In particular, for these weights our theorems are true without condition.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