Divisibility of Eigenforms, and computing a function of the j-invariant

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Goals of this talk

- Results on divisibility of eigenforms
- Computational evidence toward related conjectures
- Results on independence of L-values (Maybe)

Notations and setup

- $\Gamma = SL_2(\mathbb{Z})$ (Level 1)
- M_k = space of level 1 modular forms
- S_k = space of level 1 cuspidal modular forms
- $T_{n,k} = n^{th}$ Hecke Operator for weight k
- $T_n = n^{th}$ Hecke Operator when the weight is understood
- $T_n(x)$ = Hecke Polynomial related to T_n
- E_k = weight k Eisenstein series
- $\Delta = \sum \tau(n) q^n$ = normalized weight 12 cuspidal form

Eigenforms

Definition

A modular form $f \in M_k$ is said to be an eigenform if it is an eigenvector for all the Hecke operators $\{T_{n,k}\}_{n\in\mathbb{N}}$.

Fact

M_k has a basis of eigenforms.

Example

$M_k = \langle E_k \rangle \oplus S_k$. Both $\langle E_k \rangle$ and S_k are preserved by T_n.

Example

S_k has dimension 1 for $k \in \{12, 16, 18, 20, 22, 26\}$. In this case let $\Delta_k(z)$ be the unique normalized cusp form in S_k. In particular $\Delta_{12}(z) = \Delta(z)$.

The Question

Question

Given an eigenform h, is h divisible by another eigenform f?

Divisibility of eigenforms

Example

E_{10} is an eigenform, and $E_{10} = E_4 E_6$.

Example

Δ divides every cuspidal modular form.

Remark

If g, h are cuspidal eigenforms, and $fg = h$, then f is noncuspidal.

Example

Consider S_{28} and M_{12}. Then $S_{28} = E_4 \Delta M_{12}$, so that every $h \in S_{28}$ factors as $h = E_4 \Delta g$ for some $g \in M_{12}$.

Example

Consider $h = E_{16}\Delta - \frac{14903892}{3617} E_4 \Delta^2 - 108\sqrt{18209} E_4 \Delta^2$, which is an eigenform in S_{28}. Then one factorization of h is:

$$h = E_4 \Delta \left(E_{12} - \frac{3075516}{691} \Delta - 108\sqrt{18209} \Delta \right)$$

Results of Ghate and Duke

Theorem

The product of two eigenforms is an eigenform only in the following cases:

- $E_4^2 = E_8$
- $E_4 E_6 = E_{10}$
- $E_4 \Delta_{12} = \Delta_{16}$
- $E_4 \Delta_{16} = E_8 \Delta_{12} = \Delta_{20}$
- $E_4 \Delta_{18} = E_6 \Delta_{16} = E_{10} \Delta_{12} = \Delta_{22}$,
- $E_4 \Delta_{22} = E_6 \Delta_{20} = E_8 \Delta_{18} = E_{10} \Delta_{12} = E_{14} \Delta_{12} = \Delta_{26}$.
- $E_6 E_8 = E_4 E_{10} = E_{14}$
- $E_6 \Delta_{12} = \Delta_{18}$

Generalizations

- 2007 - Emmons and Lanphier generalized Ghate and Duke's work to an arbitrary number of eigenforms
- 2004 - Lanphier and Takloo-Bighash generalized Ghate and Duke's work to Rankin-Cohen bracket operators
- 2012 - Under the guidance of James and Xue, B. generalized divisibility to the Rankin-Cohen bracket operators (in progress)
- 2010 - Under the guidance of James and Xue, Trentacoste and myself generalized Ghate and Duke's work to nearly holomorphic modular forms.
- 2002, 2005, 2011 - Ghate, Emmons, and Johnson generalized Ghate and Duke's work to various congruence subgroups.

Corollary (B., James, Xue, 2011)

If $T_k(x)$ and $\varphi_k(x)$ are irreducible over appropriately small fields, then the only eigenforms that divide other eigenforms come from one dimensional spaces: $M_4, M_6, M_8, M_{10}, S_{12}, M_{14}, S_{16}, S_{18}, S_{20}, S_{22}$ *and* S_{26}.

Remark

Infinite classes of examples of "small" eigenforms dividing other "larger" eigenforms can be constructed.

Precise statement: One of three cases

Theorem (B., James, Xue, 2011)

If for some n, $T_n(x)$ is irreducible, then a cuspidal eigenform $h \in S_{wt(h)}$ can be factored as $h = fg$ where f is an Eisenstein series, g is a modular form if and only if $\dim(S_{wt(h)}) = \dim(S_{wt(g)})$. These are precisely the cases below.

$$
\begin{aligned}
wt(f) = 4, wt(g) &\equiv 0, 4, 6, 10 \pmod{12} \\
wt(f) = 6, wt(g) &\equiv 0, 4, 8 \pmod{12} \\
wt(f) = 8, wt(g) &\equiv 0, 6 \pmod{12} \\
wt(f) = 10, wt(g) &\equiv 0, 4 \pmod{12} \\
wt(f) = 14, wt(g) &\equiv 0 \pmod{12}
\end{aligned}
$$

Maeda's Conjecture

Conjecture (Maeda, 1997)

The Hecke algebra over $\mathbb{Q}$ of $S_m(SL_2(\mathbb{Z}))$ is simple (that is, a single number field) whose Galois closure over $\mathbb{Q}$ has Galois group isomorphic to a symmetric group $\mathcal{S}_l$ (with $l = \dim(S_m(SL_2(\mathbb{Z})))$).

Corollary

$T_n(x)$ is irreducible over all fields of small degree.

Remark

This conjecture was presented in a paper by Hide and Maeda in 1997 and was verified for weights less less than 469. Later Farmer and James verified it up to weight 2000, up to to weight 3000 by Kleinerman. $T_n(x)$ was found irreducible up to weight 4096 by Ghitza.

Conjecture on φ_k

Conjecture (2001)

Let $\varphi_k := \prod(x - j_i)$, where the product runs over all the j-zeros of $E_{wt(h)}$ except for 0 and 1728.$\left(\text{This function satisfies } \frac{E_k}{E_4^a E_6^b \Delta^c} = \varphi_k(j)\right)$ Then φ_k is irreducible will full Galois group.

Remark

In a 2001 paper by Gekeler, this conjecture was verified up to weight 700. In our own calculations we found φ_k to be irreducible up to weight 1500.

Computing φ_k

- $\frac{E_k}{E_4^a E_6^b \Delta^c}(z) = (\varphi_k(j))(z)$
- In Theory: Use the q-expansions of E_k, Δ, and j, then solve for the coefficients of $\varphi_k(j)$.
- In Practice:
  - Multiply by q^c on both sides to remove poles.
  - Do all computations modulo p, and hope φ_k mod p is irreducible. (Conjecturally there is always a prime such that φ_k is irreducible modulo p)
  - Setup the equality as a triangular system
- Choke point: Computing powers of j
  - Every power $1, ..., \frac{k}{12}$ needs to be computed
  - Runtime $O\left(n^3\right)$
  - Weight 1488, $p = 19$: 17 CPU seconds to calculate all the powers of j and combine like terms.

The relationship with L-values

Definition

Let $g = \sum a_n q^n$ and $h = \sum b_n q^n$ be cusp forms.

$$L(g,h) := L(g \times h, wt(h) - 1) = \sum_{i=1}^{\infty} \frac{a_i \overline{b_i}}{i^{wt(h)-1}}.$$

Theorem (The Rankin-Selberg Method)

Let $g = \sum a_n q^n$ and $h = \sum b_n q^n$ be cusp forms.
$\langle E_s \cdot g, h \rangle = (4\pi)^{-(s+wt(g)-1)} \Gamma(s + wt(g) - 1) \sum_{n \geq 1} \frac{a_b \overline{b_n}}{n^{s+wt(g)-1}}$

A theorem on L-values

Theorem (B., James, Xue, 2011)

Let $\{g_1, ..., g_k\}$ and $\{h_1, ..., h_l\}$ be eigenform bases for cuspidal spaces. If $T_n(x)$ is irreducible over suitably small fields for some n, then the vectors of L values given below are linearly independent over $\mathbb{C}$ if and only if $l > k$. Furthermore if $l = k$ there is a single dependency relation.

$$\left\{ \begin{bmatrix} L(g_1, h_1) \\ \vdots \\ L(g_1, h_{i-1}) \\ L(g_1, h_{i+1}) \\ \vdots \\ L(g_1, h_l) \end{bmatrix}, ..., \begin{bmatrix} L(g_k, h_1) \\ \vdots \\ L(g_k, h_{i-1}) \\ L(g_k, h_{i+1}) \\ \vdots \\ L(g_k, h_l) \end{bmatrix} \right\} \tag{1}$$

Thank You!