Recent Results on Improper Interval Graphs

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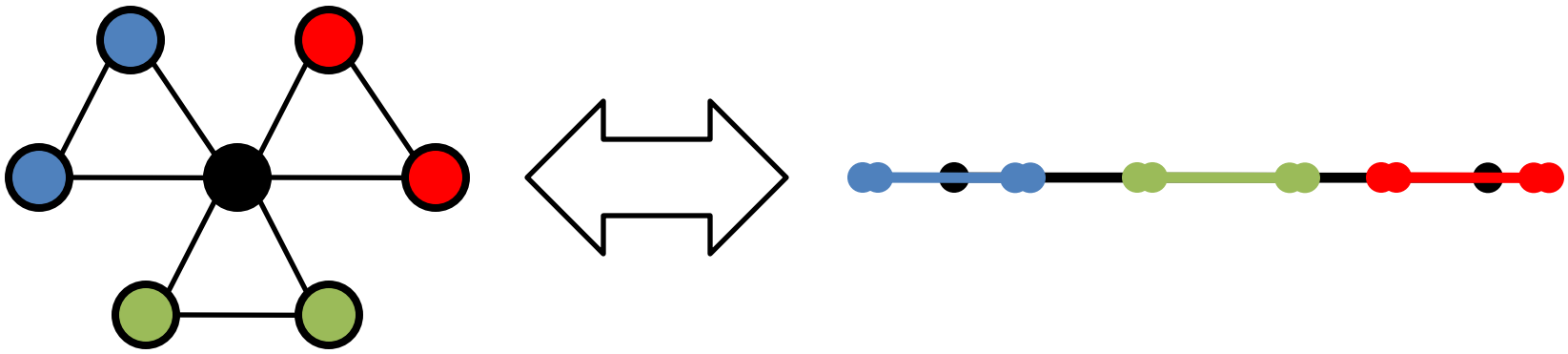
Interval Graphs

- Definition: A graph whose vertices may be represented as a set of closed intervals: where an edge occurs if and only if the corresponding intervals intersect
- (Equivalently: the intersection graph of closed intervals)



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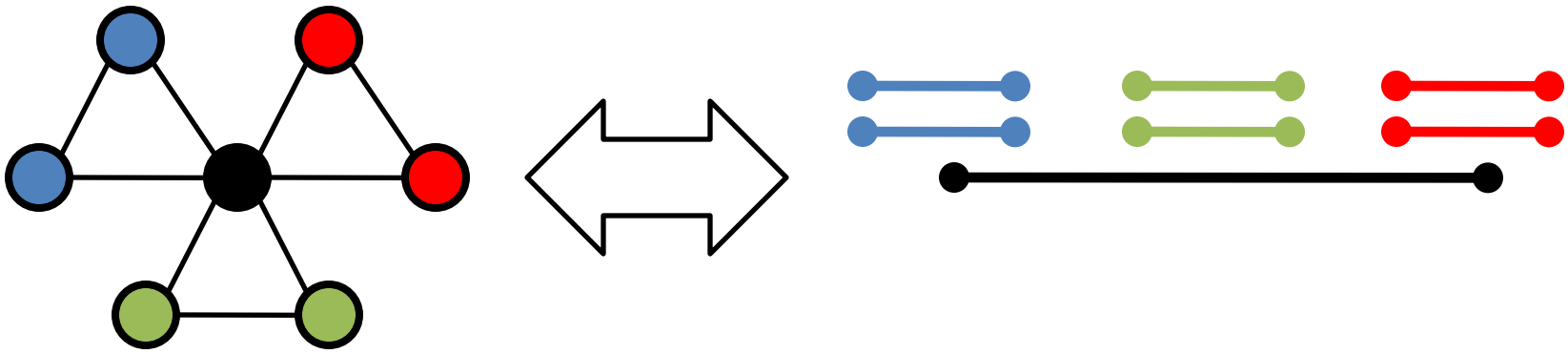
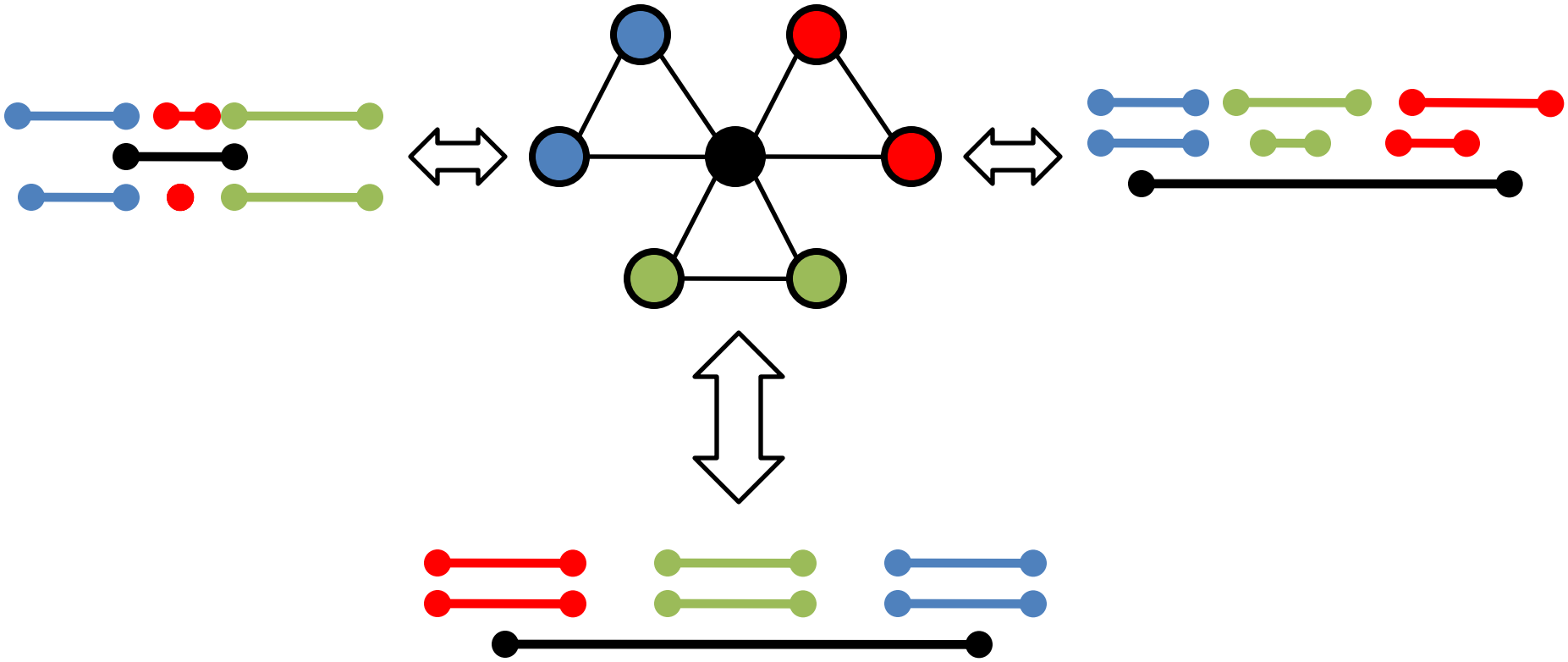
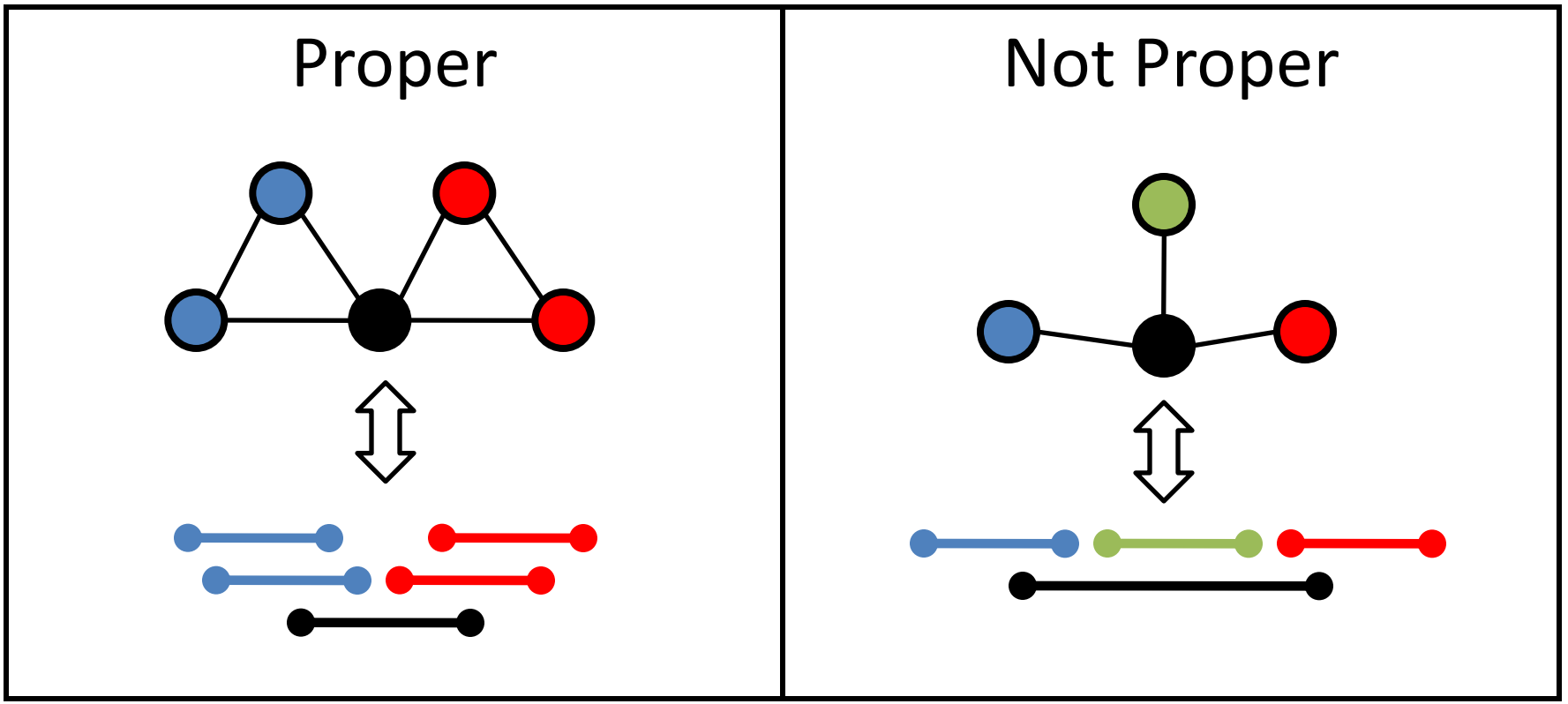


Nonuniqueness (of representation)



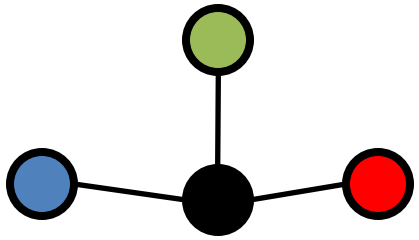
Proper Interval Graph

- Definition: An interval graph which has a representation in which no interval contains another.
- Equivalently: no interval is contained by another.



Proper Interval Graphs

- Characterized in 1969 by Fred Roberts (Right)
- Characterization: an interval graph is proper iff it has no induced $K_{1,3}$



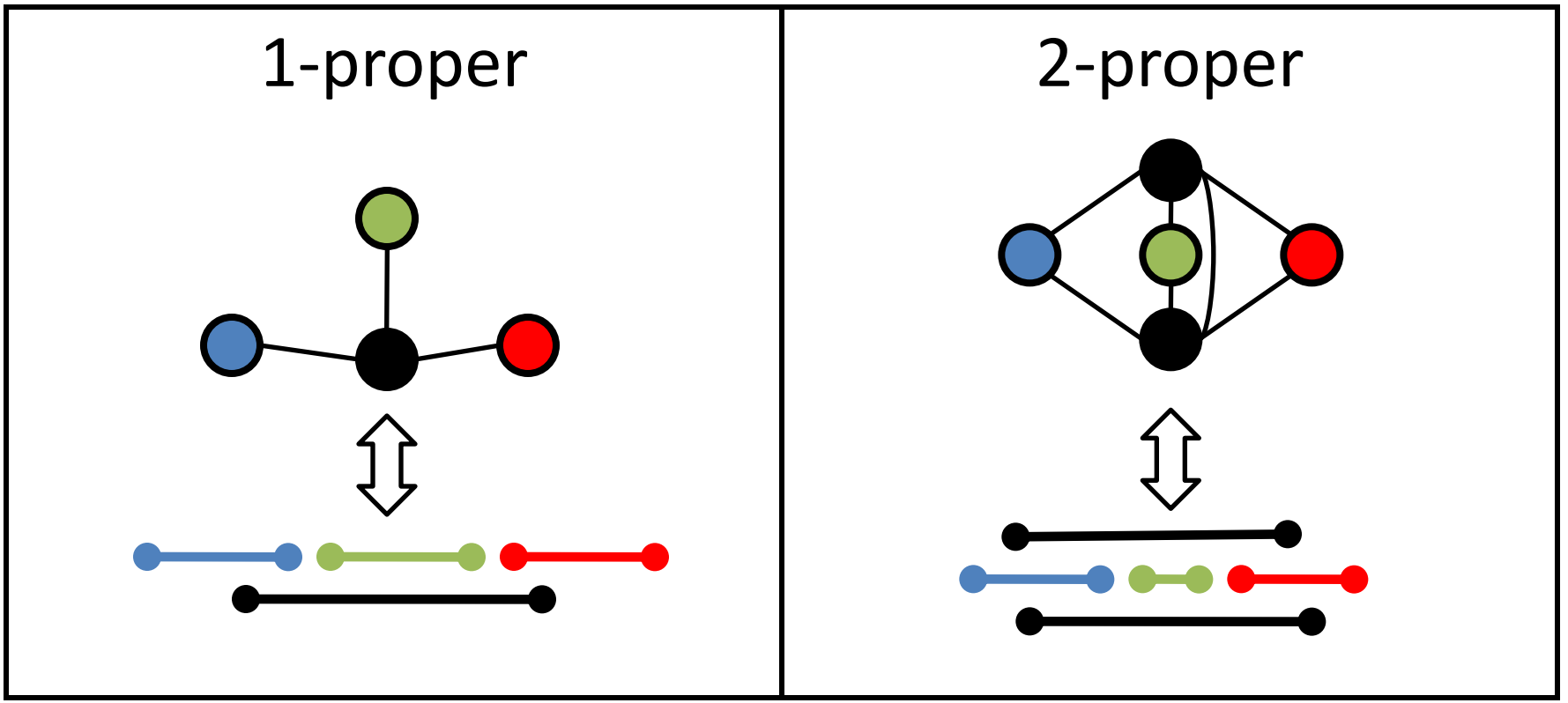
$K_{1,3}$ (aka claw, 3-star)

Two ways to generalize:

- "…contained by another"
  - Leads to q-proper interval graphs
- "…contains another"
  - Leads to p-improper interval graphs

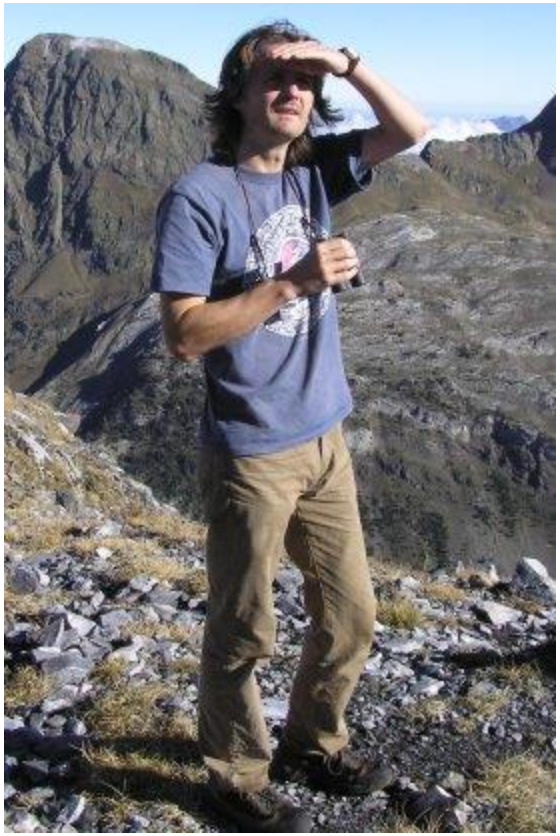
q-Proper Interval Graphs

- Definition: An interval graph that has a representation in which **no interval is contained by** more than *q* others.

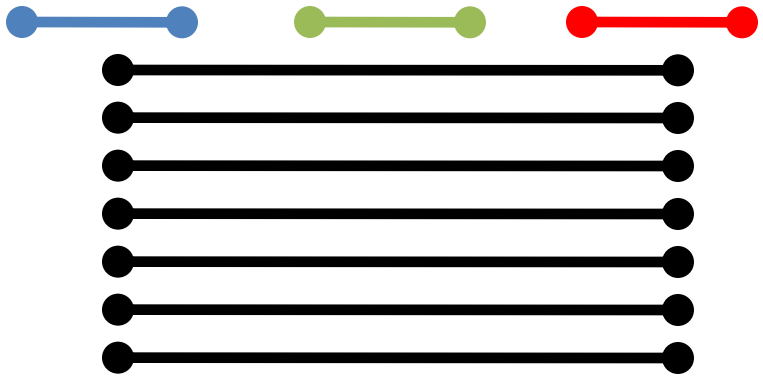


q-Proper Interval Graphs

- Characterized in 1999 A. Proskurowski (left) and J.A. Telle (right)
- Characterization: an interval graph is q-proper iff it has no induced T_{q+1}. (Tq+1 is a q+1 clique and three independent vertices, each one of which is adjacent to every vertex in the clique.)

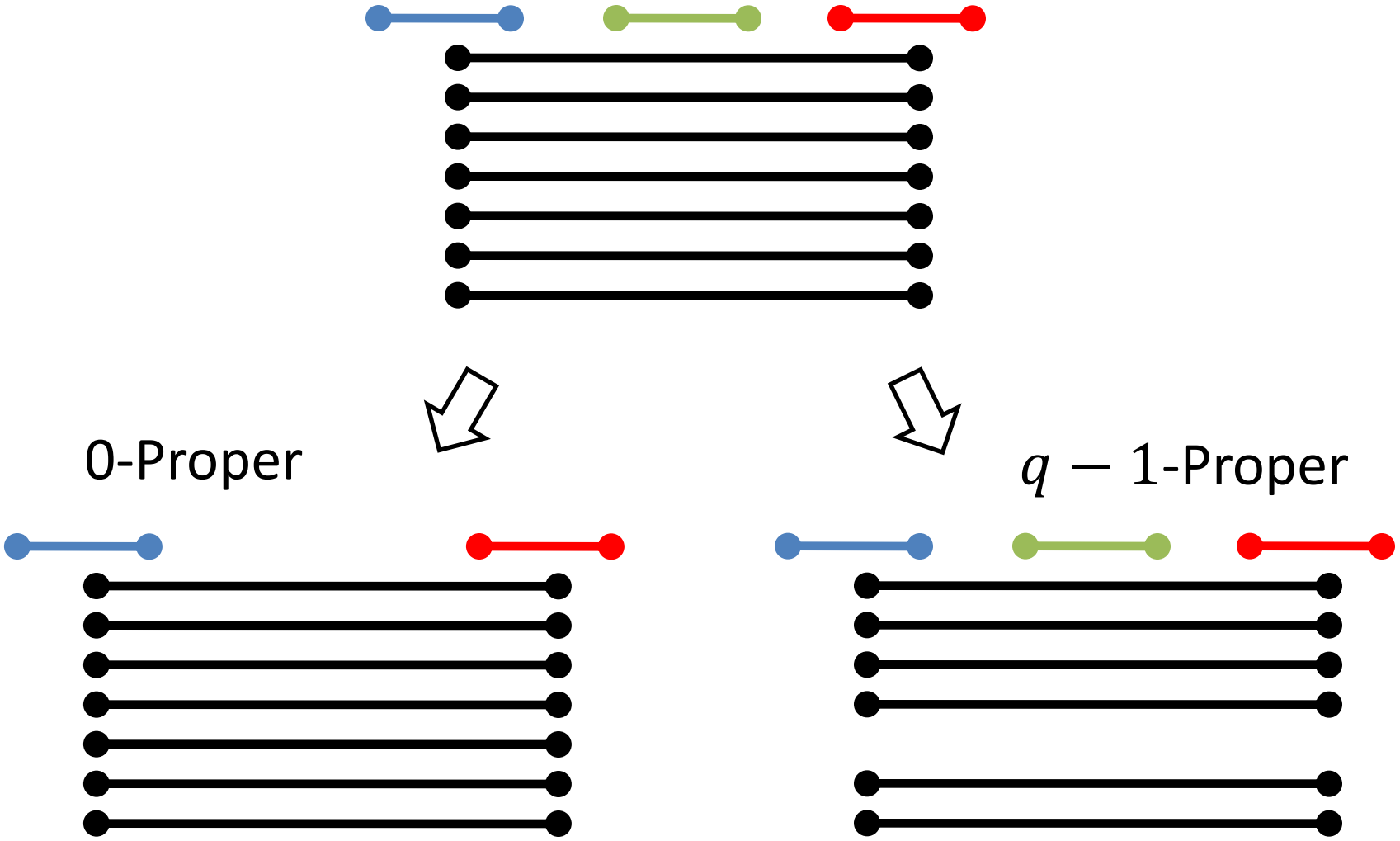


q-Proper Interval Graphs



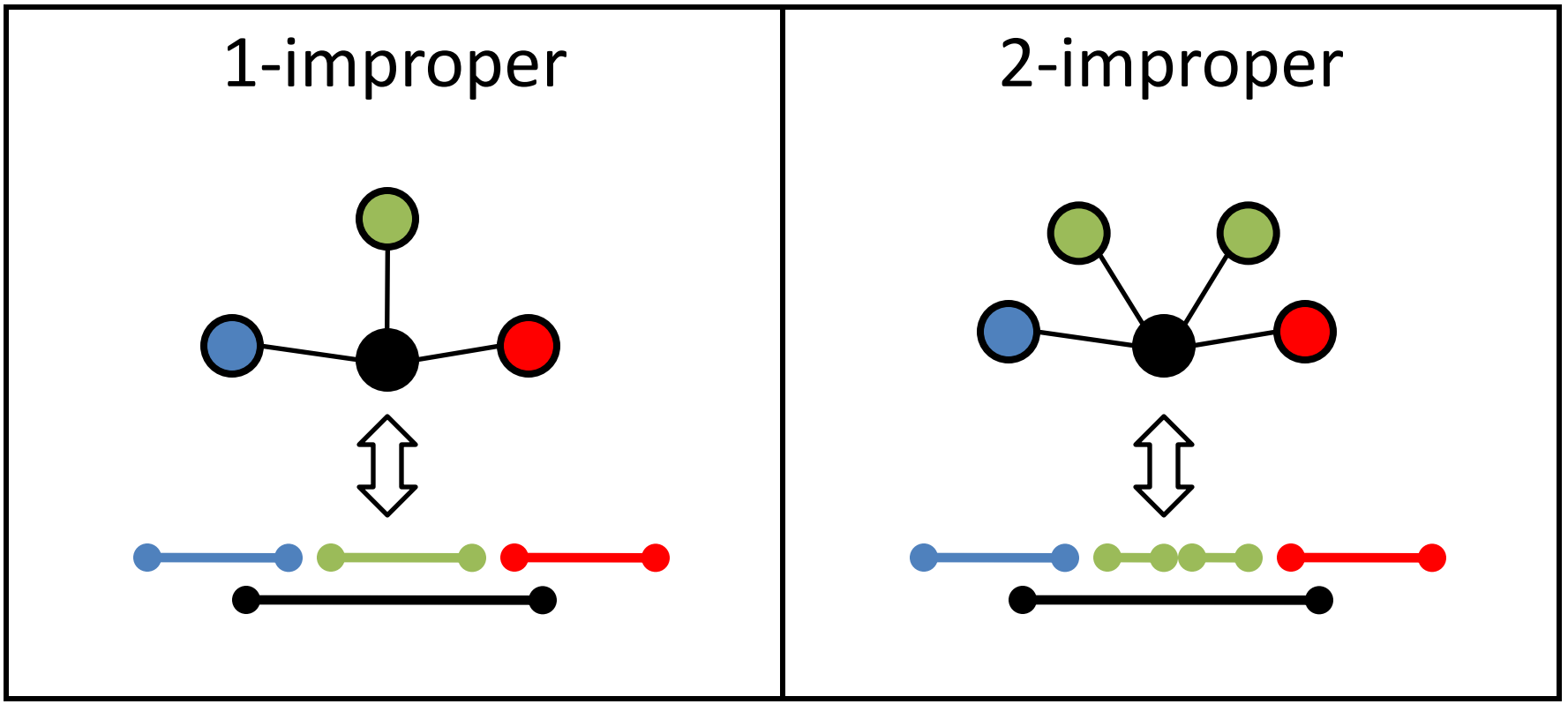
- Fairly stable: if a vertex is removed from a q-proper interval graph, it is either 0-proper or $(q-1)$-proper.

q-Proper Interval Graphs



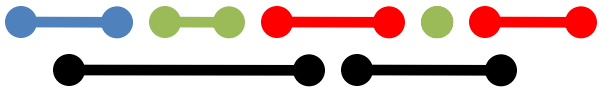
p-Improper Interval Graph

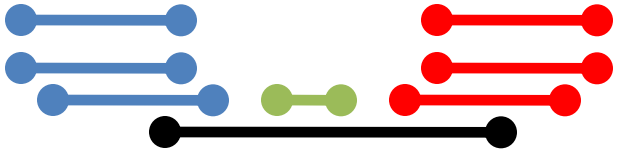
- Definition: An interval graph that has a representation in which **no interval contains** more than *p* others.

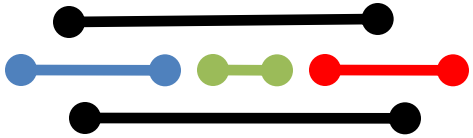


Examples

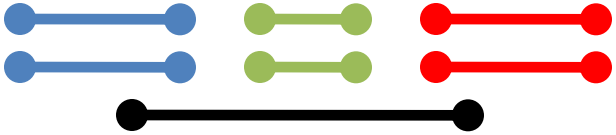
1-improper







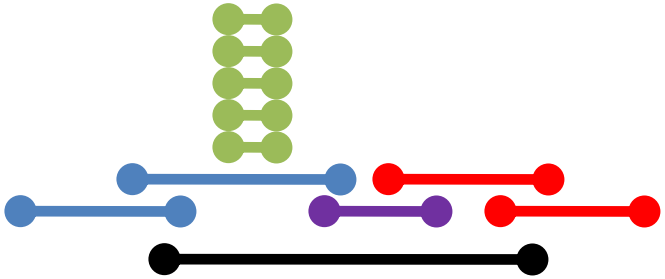
2-improper



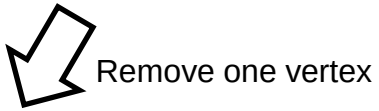
p-Improper Interval Graph

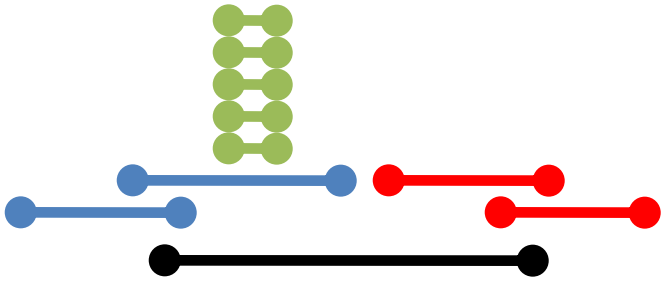
- Studied in 2008 by R. Jamison (right) and J. Beyerl.
  - Classified 1-improper interval graphs.
  - Partially classified balanced improper interval graphs
- Studied in 2012 by Wayne Wallace and J. Beyerl.
  - Fleshed out the structure of unbalanced improper interval graphs with non-exterior side components

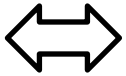
Instability

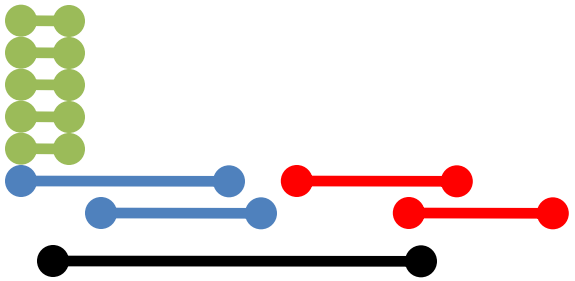


6-Improper









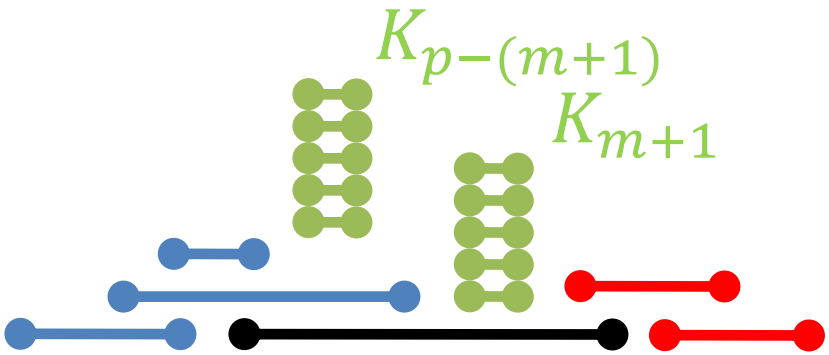
1-Improper

Instability

- Question: If one vertex is removed from a p-improper interval graph, what can we say about the resulting graph?

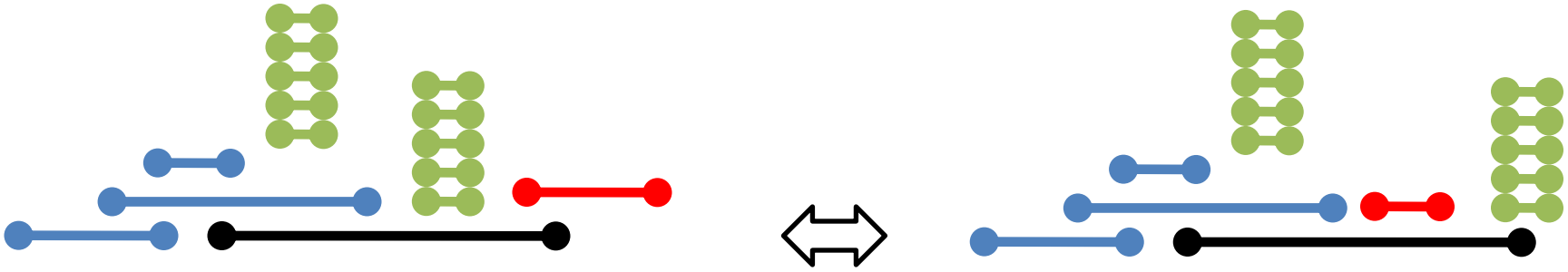
- Answer: Nothing! Or….something!

Instability



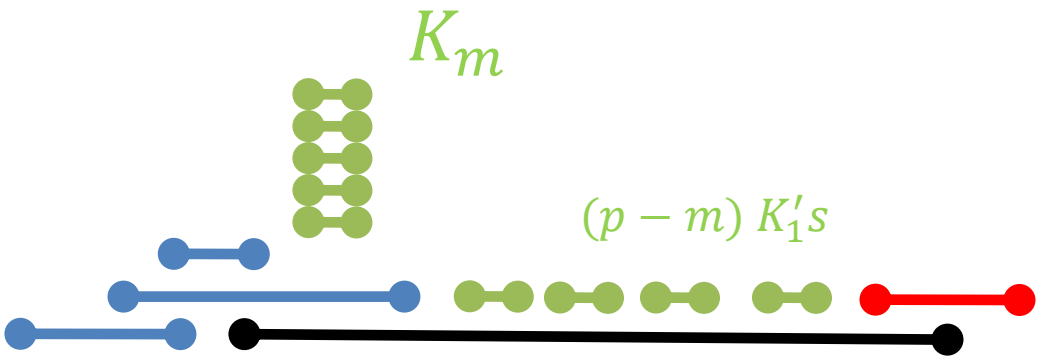
p-Improper

Remove one vertex



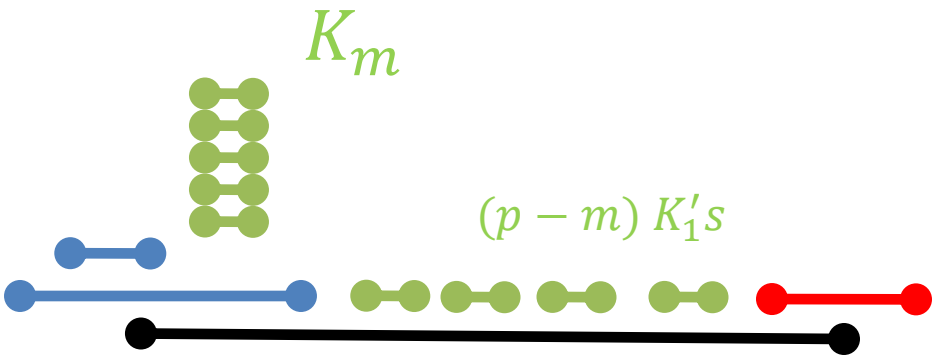
$(p-m)$-Improper

Instability



p-Improper

Remove one vertex



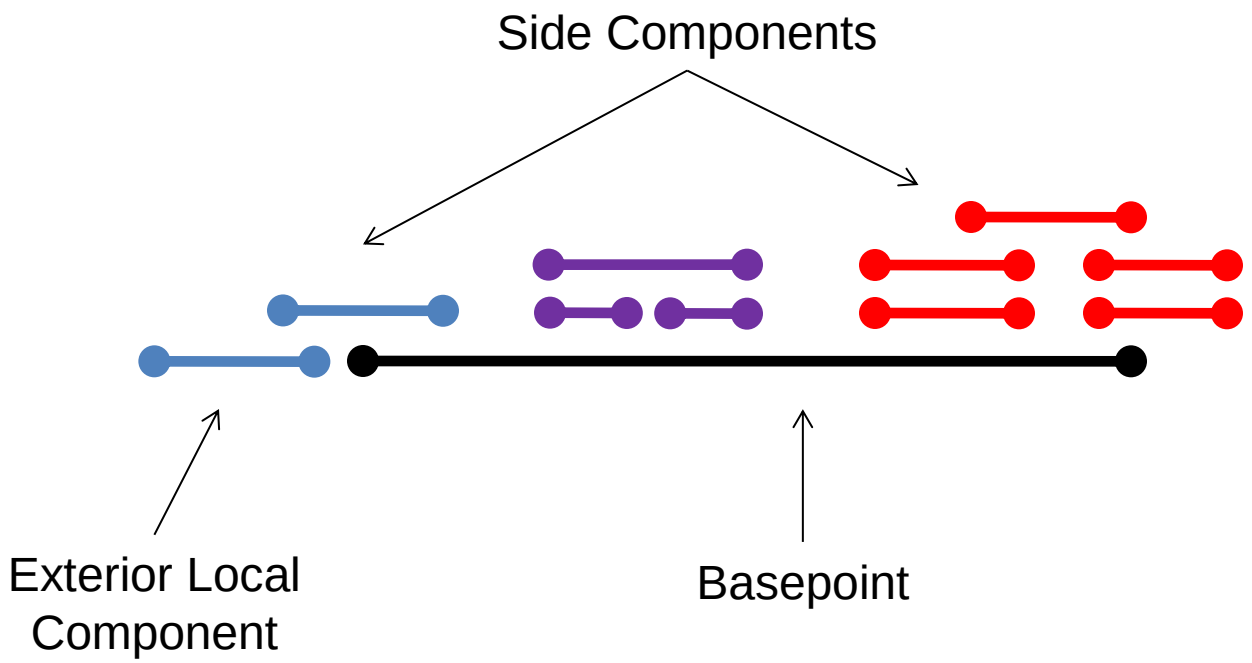
$(p-m)$-Improper

Instability

- Question: If one vertex is removed from a p-improper interval graph, what can we say about the resulting graph?
- Answer: By using an unbalanced exterior local component we can construct examples that have a $p - m$-improper subgraph after having removed a single vertex.
- Question: What if we don't allow exterior components?
- Question: What if we allow exterior components, but do not change the number of them?

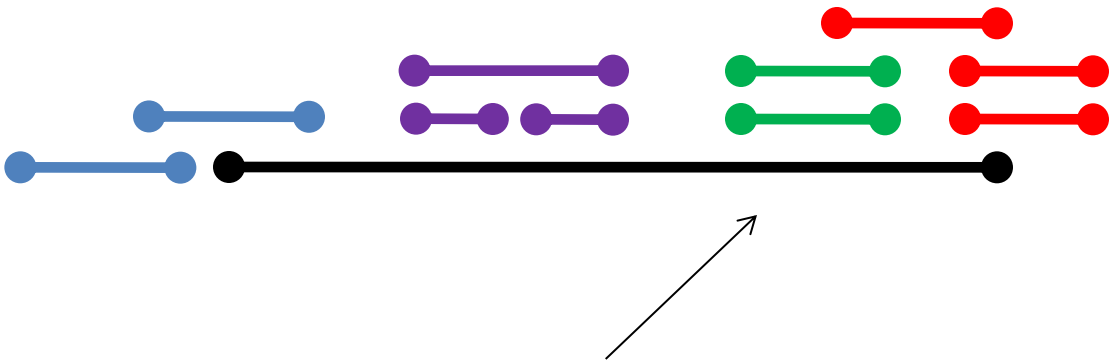
Local Components

- Given a basepoint, exterior local components are in the same place in every representation.



Balance

- A local component is considered balanced if it does not contribute to the impropriety when it is represented as a side component.



Future Questions

For me, students, or others

- Question: What kind of stability can be observed from improper interval graphs with two non-exterior local components?

- Question: What kind of stability can be observed from interval graphs with no non-exterior local components?

Thank you!