Interval Graphs with Containment Restrictions

Jeffery J. Beyerl [*] Robert E. Jamison [†‡]

Abstract

An interval graph is *proper* iff it has a representation in which no interval contains another. Fred Roberts [27] characterized the proper interval graphs as those containing no induced star $K_{1,3}$. Proskurowski and Telle [26] have studied q-proper graphs, which are interval graphs having a representation in which no interval is properly contained in more than q other intervals. Like Roberts they found that their classes of graphs where characterized, each by a single minimal forbidden subgraph. This paper initiates the study of *p-improper interval graphs* where no interval contains more than p other intervals. This paper will focus on a special case of p-improper interval graphs for which the minimal forbidden subgraphs are readily described. Even in this case, it is apparent that a very wide variety of minimal forbidden subgraphs are possible.

1 Introduction

A finite, simple graph $G = (V, E)$ is an *interval graph* iff there is an assignment $\alpha : v \longrightarrow I_v$ of vertices v of G to intervals I_v on the real line such that $vw \in E \Longleftrightarrow I_v \cap I_w \neq \emptyset$. Interval graphs appear to have first been discussed by Hajos [15]. Now classical and well-known characterizations of interval graphs were given by Lekkerkerker and Boland [23] in 1962 and Gilmore and Hoffman [7] in 1964. Extensive investigations and generalizations have since followed [2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 26, 27, 28, 29]. An interval graph is *proper* iff it has a representation in which no interval contains another. Roberts [27] introduced proper interval graphs and characterized them as interval graphs containing no $K_{1,3}$.

*Department of Mathematics, Clemson University, Clemson, SC 29634-0975 email: jbeyerl@clemson.edu

†Department of Mathematics, Clemson University, Clemson, SC 29634-0975 email: rejam@clemson.edu

‡Affiliated Professor, University of Haifa

Proskurowski and Telle [26] generalized this to *q-proper interval graphs*, graphs having an interval representation in which no interval is properly contained in more than q others.

This paper will forbid containments in the opposite direction. A *p-improper interval graph* is one having an interval representation in which no interval contains more than p other intervals. The key difference between these generalizations is that Proskurowski and Telle [26] forbid *supersets* whereas here *subsets* are forbidden.

By a *p-improper representation* we mean an interval representation with no interval containing more than p other intervals. Obviously, if G has such a representation and H is a subgraph of G, then deleting from a representation of G those intervals which correspond to vertices not in H yields a representation of H. This hereditary property guarantees that the class $\mathscr{I}_p$ of p-improper interval graphs has a minimal forbidden subgraph characterization. The class of proper interval graphs (which coincides with the class of unit interval [14]) is thus the class $\mathscr{I}_0$.

The Lekkerkerker-Boland theorem [23] says that chordless cycles and asteroidal triples form a defining class of forbidden subgraphs for the class of interval graphs. Thus we will be interested in finding minimal forbidden subgraphs within the class of interval graphs. Let $\mathscr{M}_p$ denote the set of *minimal forbidden interval subgraphs (MFISG)* for the class $\mathscr{I}_p$ of p-improper interval graphs. The *impropriety* $\mathrm{imp}(G)$ of G is the smallest p such that G has a p-improper representation. Unlike the case of q-proper interval graphs which have an essentially unique MFISG for each q, p-improper interval graphs show a great diversity of MFISGs, as we will see below. Fig. 3 shows a complete list of the MFISGs for the first class $\mathscr{I}_1$ with $p = 1$ [1]. These ten MFISGs show the breadth of possibilities right at the beginning. The star $K_{1,p+3}$ is easily seen to be a MFISG for $\mathscr{I}_p$. This is the easiest case. The next easiest case is the *balanced* case which includes three examples from Fig. 3. We will give a formal definition of balanced here and give a complete description of all MFISGs in this case.

2 Weight and Balance in Interval Graphs

Throughout this section $G = (V, E)$ will denote a finite, connected, interval graph. First we establish the notation for the central ideas of the paper. Recall that a finite, simple graph $G = (V, E)$ is an *interval graph* iff there is an assignment $\alpha : v \longrightarrow I_v$ of vertices v of G to intervals I_v on the real line such that $vw \in E \iff I_v \cap I_w \neq \emptyset$. If a representation α has been given, ℓ_v and r_v will denote the left and right endpoints, resp., of the interval I_v representing v. The *support* of a set $W \subseteq V$ of vertices in a representation α is the union of all intervals I_w where $w \in W$. The *impropriety* $\mathrm{imp}_\alpha(z)$ of a vertex z of G with respect to the representation α

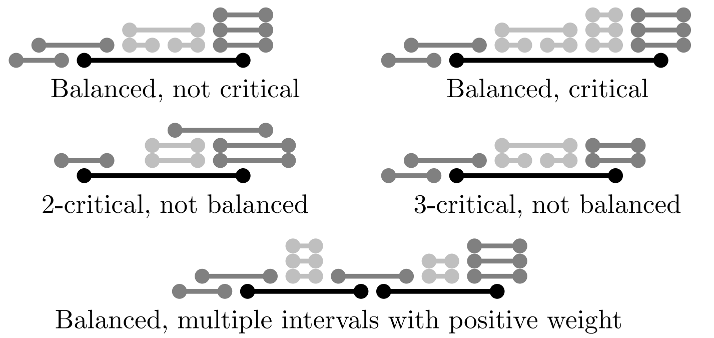

Figure 1: Illustrations of Balance and Criticality

is the number of representing intervals which lie inside I_z (not counting I_z itself). The *impropriety* $\mathrm{imp}(\alpha)$ of the representation α is the **maximum** of the improprieties $\mathrm{imp}_\alpha(z)$ over all vertices z of G. The *impropriety* $\mathrm{imp}(G)$ of G is the **minimum** of $\mathrm{imp}(\alpha)$ over all representations. A representation which minimizes the impropriety will be called a *minimal representation.* That is, a representation α is minimal iff $\mathrm{imp}(\alpha) = \mathrm{imp}(G)$.

For $z \in V$, a component of $G \setminus \{z\}$ will be called a *local component* at z (or more simply, just a *component at z*). A local component is *exterior* iff it contains a vertex not adjacent to z.

Lemma 2.1 *A vertex z in an interval graph can have at most two exterior (local) components.*

Proof. If there are three exterior components C_1, C_2, C_3, choose vertices a_1, a_2, and a_3 at distance two to z with $a_i \in C_i$. Then a_1, a_2, and a_3 form an asteroidal triple, which by [23] is forbidden in an interval graph. ■

A vertex z of G is *type k* iff v has exactly k exterior components. By Lemma 2.1 k can take on only three values: 0, 1, or 2.

We now introduce a quantity which provides a lower bound on — and sometimes an exact value for — the impropriety. Suppose z has n local components $C_1, C_2, C_3, ..., C_n$. The *weight* $\mathrm{wt}(z)$ of z is the sum of the $n-2$ smallest orders of the **non-exterior** local components. The *weight* $\mathrm{wt}(G)$ of G is the maximum of the weights of its vertices. Note that the weight is defined in terms of the graph G directly and does not depend on any particular representation. Impropriety, on the other hand, is defined in terms of representations of G.

Let us consider some examples of this somewhat confusing concept. Let X_1 and X_2 denote generic exterior components. Let A, B, C, D, F denote local components with orders $A = 5, B = 5, C = 5, D = 4, F = 2$.

Suppose the local components at z are	Excluded Loc Comp	The counted orders are	Weight
X_1, X_2, A, B	X_1, X_2	5+5	10
X_1, X_2, C, F	X_1, X_2	5+2	7
X_1, A, B	X_1, A	5	5
X_1, C, F	X_1, C	2	2
A, B, C, D, F	A, B	5+4+2	11
C, D, F	C, D	2	2

Suppose the local components at z are	Nr. of Comp.	$n-2$ smallest Non-Exterior Local Comp.	Weight
X_1, X_2, A, B, C, D, F	$n = 7$	A, B, C, D, F	5+5+5+4+2 = 21
X_1, X_2, C, F	$n = 4$	C, F	5+2 = 7
X_1, X_2	$n = 2$	none	0
X_1, A, B, C	$n = 4$	B, C	5+5 = 10
X_1, C, F	$n = 3$	F	2
A, B, C, D, F	$n = 5$	C, D, F	5+4+2 = 11
C, D, F	$n = 3$	F	2
C, F	$n = 2$	none	0
D	$n = 1$	none	0

Theorem 2.2 *If z is any vertex of an interval graph G, the impropriety of G is at least the weight of z.*

Proof. Consider any interval representation $\alpha : v \to I_v$ of G. The supports of the local components are themselves disjoint intervals which lie left to right along the line. Say the local components in this ordering are $A_1, A_2, A_3, ..., A_n$. Then the components $A_2, A_3, ..., A_{n-1}$ must have supports entirely inside I_z. Thus each of these local components lies in the neighborhood of z. Hence if there are exterior components they must be A_1 or A_n, or both. In any case, the $n-2$ components $A_2, A_3, ..., A_{n-1}$ are not exterior and thus the sum of their orders is at least $\mathrm{wt}(z)$. Thus we have shown that in any representation, I_z contains at least $\mathrm{wt}(z)$ other intervals. Thus the impropriety of G is at least $\mathrm{wt}(z)$, as desired. ■

Corollary 2.3 *For any interval graph G, $imp(G) \geq wt(G)$.*

G is *balanced* iff $\mathrm{wt}(G) = \mathrm{imp}(G)$. If G is balanced, a vertex z such that $\mathrm{wt}(z) = \mathrm{imp}(G)$ is a *basepoint* of G. Equivalently, z is a basepoint iff G is

11-improper 2-improper

Figure 2: Example of Instability in the Impropriety

balanced and z has maximum weight. Notice that a basepoint must have at least three local components since a vertex with only one or two local components has weight 0.

Theorem 2.4 *If G is a connected, interval graph, then the vertices of positive weight induce a disjoint union of paths.*

Proof. Suppose α is a representation of G and suppose $I_v \subseteq I_w$. Then every neighbor of v is also a neighbor of w. Hence in $G \setminus \{v\}$, all the neighbors of v are still connected. Hence v has only one local component, so $\mathrm{wt}(v) = 0$.

Now suppose some vertex v with $\mathrm{wt}(v) > 0$ has three neighbors a, b, c also with positive weight. Suppose α is a representation of G. We saw above that none of the intervals I_a, I_b, or I_c can be contained in I_v. Thus two of these intervals must exit I_v on the same side. Say, I_a and I_b exits I_v through the right end point r_v of I_v. Without loss of generality, assume $\ell_a \leq \ell_b$. Since no interval of positive weight can contain another, $r_a \leq r_b$ is forced. Thus $I_a \subseteq I_v \cup I_b$. But this means that any neighbor of a must be a neighbor of either v or b. Since v and b are adjacent, it follows that a has only one local component, and hence has $\mathrm{wt}(a) = 0$, a contradiction.

■

3 p-critical Interval Graphs

An interval graph G is *p-critical* with respect to impropriety iff G has impropriety p but every proper induced subgraph of G has impropriety strictly less than p. Note that the concept of p-critical only makes sense for $p > 0$. Clearly, a $p + 1$-critical graph is a MFISG for the class $\mathscr{I}_p$ of p-improper interval graphs. The converse is not so clear. Fig. 2 gives an example where the impropriety changes drastically with the removal of a single vertex.

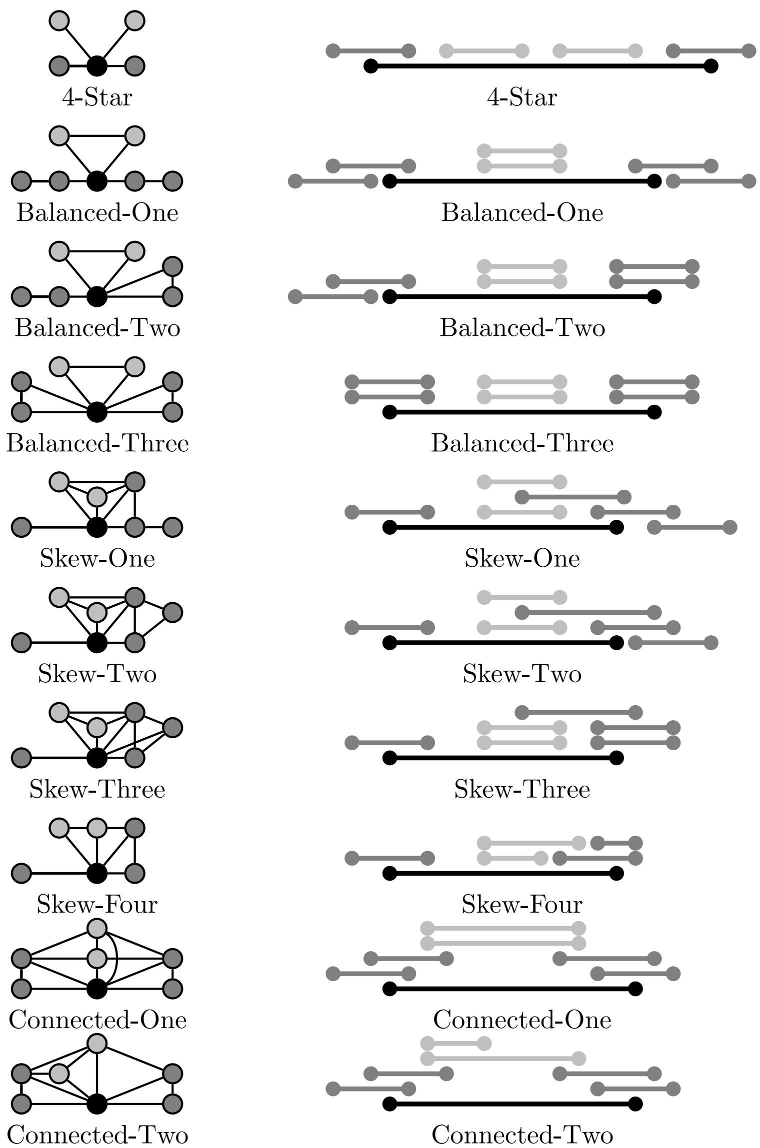

Figure 3: Minimal Forbidden Subgraphs for $\mathscr{I}_1$

Theorem 3.1 *Let z be a vertex of maximum weight in a balanced p-critical graph G. If C is an exterior local component at z, then C consists of exactly two vertices.*

Proof. Let v be a vertex in C at distance 2 from z, and let w be a common neighbor of v and z. Let H be the graph obtained from G by deleting all vertices of C other than v and w. The local components at z in H are the same as in G except that C is replaced by $\{v, w\}$. Hence the $n-2$ smallest non-exterior local components at z in H are the same as in G. Thus the weight of z in H is the same as the weight of z in G. Since G is balanced and C contains vertices other than v and w, then H is a proper induced subgraph of G and hence has a strictly smaller impropriety. Thus we have

$$\mathrm{wt}_H(z) \;\leq\; \mathrm{imp}(H) \;<\; \mathrm{imp}(G) \;=\; \mathrm{wt}_G(z) \;=\; \mathrm{wt}_H(z),$$

a contradiction. Hence C must be just $\{v, w\}$ as desired. ∎

Theorem 3.2 *If G is balanced and p-critical, then G has exactly one basepoint.*

Proof. Suppose y and z are distinct basepoints. Because G is connected, y must belong to some local component C of z. This component must also contain all p of the vertices whose intervals are contained in I_y. Since G is balanced and $p \geq 1$, any basepoint for G must have at least three local components and hence at least three neighbors. Thus since exterior components contain only 2 vertices by Lemma 3.1, C cannot be exterior. Dually, z is contained in a local component D at y, which, dually, is not exterior. Since z has at least three local components, there is a local component A at z which is disjoint from C. That is, z is adjacent to vertices not adjacent to y. But that means, D is an exterior component at y, a contradiction. ∎

Theorem 3.3 *Suppose G is balanced and p-critical. Let z be the basepoint of G.*

a) If there is at most one exterior component at z, then there are at least two local components at z which are cliques and have maximum order among the local components.

b) If there is no exterior component at z, then there are at least three local components at z which are cliques and have maximum order among the local components.

Proof. Select a minimal representation α of G. As in the proof of Theorem 2.2, look at the supports of the local components. These are disjoint intervals, ordered from left to right. Call the leftmost and rightmost components the *side components*. The other components are *inner components*. By hypothesis, at most one local component can be exterior, so at least one of the side components is non-exterior. Call such a component A. For concreteness, suppose A in on the right side. The weight is determined by adding the orders of the non-side components. Since α is minimal and G is balanced, the impropriety equals the sum of the orders of the inner components. Hence A cannot contribute to the impropriety. Now consider $v \in A$. Since A is not exterior $I_v \cap I_z \neq \emptyset$. Thus $\ell_v \leq r_z$. Since A does not contribute to the impropriety, I_v is not contained in I_z. Since A is on the right side, this says $r_z < r_v$. Combining these inequalities, we find $r_z \in I_v$ for all $v \in A$, so A is a clique.

If there are no exterior components, the above argument shows that both the right and left side components must be cliques.

Now let A and B the side components. If one of these is exterior, by symmetry it may be assumed to be B. Thus from the way that weight is defined and because α is a minimal representation, it follows that A is a component of maximum order. If there are no exterior components, then, by symmetry, A can be assumed to have order greater than or equal to B. Thus in either case, we can assume that A is local component of maximum order.

Suppose $x \in A$. Since G is p-critical, it follows that removing x will decrease the impropriety. That is, we need to find a representation of $G \setminus \{x\}$ which has a lower impropriety. Any representation consists of the local components strung out in some order along I_z. Rearranging the inner components among themselves or changing the way they are represented will not decrease the number of intervals contained in I_z. Thus some inner component must trade places with one of the two side components. If exchanging an inner component for B has a helpful effect, this helpful effect would be present even if x is left in A. That is, this move could be used to give a representation for G with a smaller impropriety, contrary to the minimality of α. Thus the essential move is exchanging an inner component C for $A \setminus \{x\}$.

Suppose A has order m and C has order n. This exchange increases the number of intervals contained in I_z by $m-1$ and decreases it by at most n. The inequality here arises if C is not a clique, so that some of its intervals must intersect I_z while avoiding other intervals from C. This would force some intervals arising from C to be wholly contained in I_z.

Now $n \leq m$ since A has maximum order. The decrease d in impropriety thus satisfies $d = n - (m-1) \leq 1$. Conversely, $d \geq 1$ since G is p-critical. Thus $n - (m-1) = d = 1$, so $n = m$. And this occurs iff all intervals in C can be moved out of I_z — that is, C is a clique.

Thus we have shown that there must be one side component A that has maximum order and is a clique. Moreover, there must be an inner component C that has maximum order and is a clique. If the type is 0, then B exists and, as shown above, B must be a clique. If it is not of maximum order, interchanging B and C would reduce the impropriety of the representation, contrary to the assumption that α is maximal. ■

4 Construction of Balanced Interval Graphs

Let z denote an isolated vertex. Let $\mathcal{H} := H_1, H_2, H_3, \dots, H_n$ denote a sequence of interval graphs. Let $\mathsf{BAL}_0(\mathcal{H})$ denote the join of z with the disjoint union of the H_i. That is, z is made adjacent to all vertices in all of the H_i. This is clearly an interval graph: represent z by a long interval and draw representations of the H_i in disjoint subintervals of this long interval. A *pendant* P_3 at z is a path xyz such that y is adjacent only to z and x and x is adjacent only to y. If in addition the maximum order of the $\mathcal{H}_i$ is at least 2, $\mathsf{BAL}_k(\mathcal{H})$ denotes $\mathsf{BAL}_0(\mathcal{H})$ with $k \geq 1$ pendant P_3's attached to z.

Theorem 4.1 *A graph G is p-critical and balanced iff*

a) G is isomorphic to $\mathsf{BAL}_0(\mathcal{H})$ where three of the H_i having maximum order are cliques;

b) G is isomorphic to $\mathsf{BAL}_1(\mathcal{H})$ where two of the H_i having maximum order are cliques;

c) G is isomorphic to $\mathsf{BAL}_2(\mathcal{H})$ for interval graphs H_i.

Proof. If G is p-critical and balanced, then by Theorems 3.1 and 3.3, G has the form specified above. For the converse, suppose G has the form specified above. It is convenient to assume that $\mathcal{H} := H_1, H_2, H_3, \dots, H_n$ is ordered so that $|H_i| \leq |H_{i+1}|$ and among the H_i of maximum order, the cliques come last.

If $k = 2$, construct a representation α of $G = \mathsf{BAL}_2(\mathcal{H})$ by putting the two pendant P_3's at either ends of a long interval I_z for z. Represent the H_i inside smaller subintervals of I_z. The weight of z in $G = \mathsf{BAL}_2(\mathcal{H})$ is clearly $\Sigma := \sum_{i=1}^{n} |H_i|$. This is also the impropriety of z in the representation α. Thus $\Sigma = \mathrm{wt}(z) \leq \mathrm{wt}(G) \leq \mathrm{imp}(G) \leq \mathrm{imp}(\alpha) = \Sigma$. Therefore, $\mathrm{wt}(G) = \mathrm{imp}(G)$, so BAL_2-graphs are balanced.

To show BAL_2-graphs are critical, it suffices to show that if any interval from the representation α is removed, then the remaining intervals can be rearranged to reduce the impropriety. An inner interval contributes directly to the impropriety, so its removal reduces the impropriety. Thus consider a pendant P_3 xyz. If y is removed, then H_n can be moved to where I_y was.

This decreases the impropriety by $|H_n|$. If x is removed, then the interval I_y for y can be exchanged for H_n. This reduces the impropriety by $|H_n|-1$. But $|H_n|$ is maximal, and by definition of BAL_2, there is a local component with at least two vertices. Thus $|H_n|-1>0$, so the impropriety does go down.

If $k=1$, put the pendant P_3 to the left of a long interval I_z for z. Put small intervals for H_n, all containing the right endpoint of I_z. As before, represent the remaining H_i in smaller intervals contained in I_z. The weight of z in G is $\sum_{i=1}^{n-1}|H_i|$. This is again $\mathrm{imp}(\alpha)$. As in the case $k=2$, this implies BAL_1-graphs are balanced.

In showing criticality, pendant P_3's and inner intervals can be treated the same way as for $k=2$. If a vertex is removed from H_n, then we can exchange H_n for H_{n-1} which is an interior clique of the same order as H_n by hypothesis. This reduces the impropriety by 1.

If $k=0$, H_n and H_{n-1} go on the ends. Removing an interior interval obviously reduces the impropriety as before. If an interval is removed from one of the end clique components, it can be exchanged for H_{n-2}. ■

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