Research Statement

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Philosophy

Research is a duty of all enlightened people that have the ability to advance the body of knowledge. My dissertation research has been in number theory, whose details are in the proceeding sections. In particular it concerns the factorization of a very nice class of functions called eigenforms, relations to L-values, and nearly holomorphic eigenforms.

Research is the chance to increase the knowledge of the human race. This is a truly worthwhile endeavor. Oh how great it is to begin with what was known and to end with something completely new!

I spend my time thinking about what properties certain objects may have, or how they interact with other objects. When conversing with other mathematicians such as after a seminar or during a conference there is also a collaborative aspect of “bouncing” one’s research off another to obtain different viewpoints.

Background

Recently my research has been in the field of modular forms. Modular forms are a theoretical branch of number theory with numerous applications. One of the most notable examples is Andrew Wiles’s proof of Fermat’s Last Theorem which showed that all elliptic curves come from modular forms.

In particular a modular form of level 1 is a holomorphic function $f(z)$ on the upper half plane such that the following modular equation holds true:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. The integer k is called the weight of the modular form. Essentially this says that f transforms in a particularly nice way under the standard fractional linear transformation of $SL_2(\mathbb{Z})$ on the upper half plane.

The space of all modular forms gives an algebra graded by weight. A class of linear operators of particular note are the Hecke Operators, $\{T_{n,k}\}$, which act on forms of weight k, and are given by

$$\left(T_{n,k}(f)\right)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right).$$

In particular for a fixed weight k there is an infinite class of Hecke Operators $\{T_{n,k}\}_{n\in\mathbb{N}}$ that act on modular forms of weight k.

Eigenforms

Some of my research involves looking at a certain class of particularly nice modular forms called eigenforms. An eigenform f of weight k is a modular form which is an eigenvector for all the Hecke Operators $\{T_{n,k}\}_{n\in\mathbb{N}}$ simultaneously. This is adding additional requirements to the modular form yielding them to be among an elite class of functions.

A first natural question to ask is whether or not the product of two eigenforms is again an eigenform. Almost always the answer to this question is no.

I have taken this idea further and asked a more general question about the factorization of eigenforms: when does an eigenform have another eigenform as a factor? For instance if f and g are eigenforms, is there a modular form h such that $f = g \cdot h$? The answer here is that only very "small" eigenforms should be factors of eigenforms, albeit every eigenform has at least one eigenform factor: either an Eisenstein series or Δ, the 24th power of the Dedekind Eta function. For example, the following is the first nontrivial such equation:

$$\underbrace{E_{16}\Delta - \left(\frac{14903892}{3617} - 108\sqrt{18209}\right)E_4\Delta^2}_{\text{Eigenform}} = \underbrace{E_4\Delta}_{\substack{\text{"Small"}\\ \text{Eigenform}}} \times \underbrace{\left(E_{12} - \left(\frac{3075516}{691} + 108\sqrt{18209}\right)\Delta\right)}_{\text{Non Eigenform Modular Form}}$$

This is made more explicit my most recent paper, *Divisibility of an Eigenform by another Eigenform* available on my website < http://eweb.furman.edu/~jbeyerl>.

The Nearly Holomorphic Setting

Another direction in my research is to move to other settings such as nearly holomorphic modular forms, a special class of weakly holomorphic modular forms. This is a similar setting in which the same types of questions can be asked. One must be careful to define the properties in a manner such that the situation is analogous. The answers may take more work and sometimes the proof techniques and/or answers are not the same.

In particular modular forms satisfy two important properties: the modular equation and holomorphicity. Sadly the derivative $\frac{df}{dz}$ of a modular form f is not a modular form because the modular equation is not satisfied. However, by adding a nonholomorphic correction term, $\frac{-k}{4\pi\,\mathrm{Im}(z)}$, one can recover the modular equation. The operator that differentiates a modular form and adds this correction term is called the Maass-Shimura Operator:

$$\delta := \frac{1}{2\pi i}\left(\frac{k}{2i\,\mathrm{Im}(z)} + \frac{\partial}{\partial z}\right)$$

A general nearly holomorphic modular form is a linear combination of forms obtained in this manner:

$$\delta^{(r_1)}f_1 + \cdots + \delta^{(r_m)}f_m$$

for modular forms $f_1, \ldots, f_m$ and integers $r_1, \ldots, r_m \in \mathbb{N}$ such that each term has the same weight. The collection of all such forms is called the space of nearly-holomorphic modular forms.

Hecke Operators may be defined in a manner similar to the holomorphic setting and eigenforms again come into play. One may then ask if the product of nearly holomorphic eigenforms is again an eigenform. In the paper *Products of Nearly Holomorphic Eigenforms* I proved exactly when the product of two nearly holomorphic eigenforms is again a nearly holomorphic eigenform. This special situation happens only 18 times and never nontrivially.

L-Values

The Riemann Zeta Function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, plays an important role in much of advanced mathematics and physics. It is most notably famous in its role in the Riemann Hypothesis. *L* functions generalize the Riemann Zeta Function and are also quite important creatures in mathematics. In particular they play an important role in the Birch and Swinnerton-Dyer Conjecture, one of the lesser known Clay Institute Millennium Problems.

The L-functions my research deals with are a more general L-function arising from modular forms, say, $f(z) = \sum_{n=0}^{\infty} a_n q^n$ and $g(z) = \sum_{n=0}^{\infty} b_n q^n$ where $q = e^{2\pi i z}$ and are given by

$$L(f \times g, s) := \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^s}.$$

An L-value is an L function evaluated at a specific value of s, although their values are often very difficult to determine exactly. In particular it is rare to know much about the independence of *L*-values. Surprisingly my work relates the independence of L-values to the factorizations of eigenforms. For example there is a factorization $f = gh$ as above precisely when there is a dependence relation among certain L values. This yields insight to both fields. For more information see my paper *Divisibility of an Eigenform by another Eigenform.*

Experimental Mathematics

On the experimental side of mathematics some of my work on the factorization of Eigenforms is related to Maeda's conjecture. This is a modern conjecture regarding the structure of Hecke Operators and their associated algebra. It states that the Hecke Algebra is simple and its Galois closure over $\mathbb{Q}$ has the full Galois group S_n. My work lends yet more support for this and a related conjecture concerning the irreducibility of $\varphi_k = \prod(x - j_i)$ where the product runs over all the j-zeros of E_k except for 0 and 1728. It satisfies:

$$\frac{E_k}{E_4^a E_6^b \Delta^c} = \varphi_k(j)$$

Future Directions and Student Projects

In my next research project, I would like to extend the work I have done on the factorization of eigenforms to answer similar questions. In particular, examples of projects include:

- When does a nearly holomorphic eigenform divide another nearly holomorphic eigenform?
- When does a level n eigenform divide another level n eigenform?
- When does the Rankin-Cohen Bracket Operator give an eigenform?
- How will these ideas generalize to weakly holomorphic modular forms?

There are also several projects here that would be appropriate for students. In particular, the similarity between the various settings allows this to be a somewhat accessible topic once the student is familiar with modular forms. I would love to be able to guide students to take what I have done and go further with it. Examples of student projects include:

- When is the Rankin-Cohen Bracket Operator is zero?
- When does a level p eigenform divide another level p eigenform?
- When is the product of many nearly holomorphic eigenforms an eigenform?
- Compute the factorizations of Hecke Polynomials to add even more support for Maeda's conjecture.